Outline:

1 Witsenhausen's counterexample.

2 Signaling strategies.

3 The infinite-length Witsenhausen counterexample.

4 The finite-dimensional problem.

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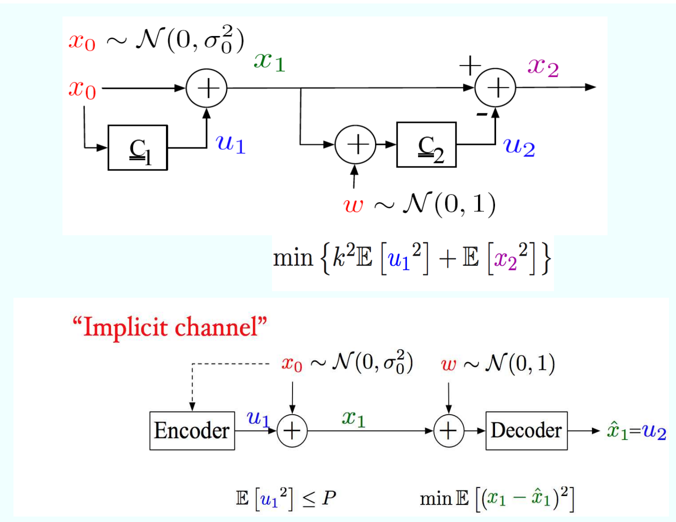
"The finite-dimensional Witsenhausen Counterexample"
*Control over Communication Channels (ConCom)*, 2009.

Slides available: www.eecs.berkeley.edu/∼pulkit/ConCom09Slides.pdf
This Handout: .../∼pulkit/ConCom09Handout.pdf
Further discussion and references can be found in the paper.

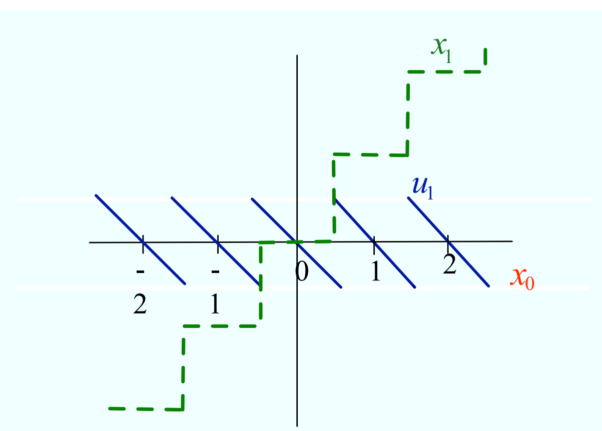
# 1 The Witsenhausen counterexample




The Witsenhausen counterexample [Witsenhausen'68] is a distributed Linear-Quadratic-Gaussian problem for which nonlinear control laws outperform the optimal linear control law. The optimal control law is still elusive — and the non-Gaussian relaxation of the problem is known to be NP-complete [Papadimitriou and Tsitsiklis '86].

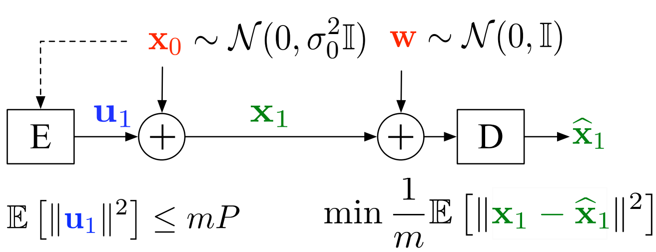
The counterexample has an implicit channel that makes it hard. The first controller modifies the initial state and attempts to communicate the *modified state* to the second controller through a noisy channel. Solving the problem of minimizing MMSE error in estimating $\widehat{x}_1$ with an average power constrained $u_1$ is an equivalent formulation.

# 2 Signaling strategies over the implicit channel



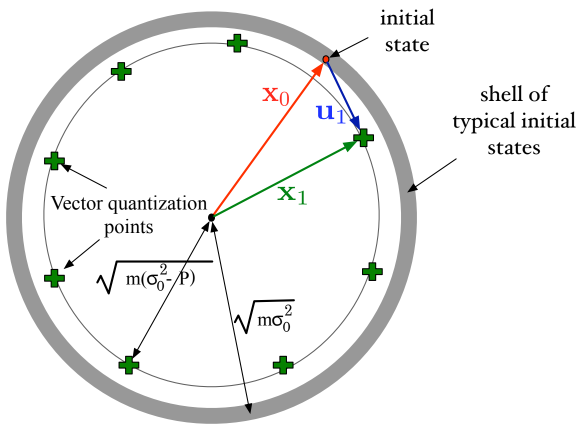

[Mitter and Sahai, 99] used quantization-based signaling strategies that beat the optimal linear strategy by a factor that can be arbitrarily large, depending on the problem parameters. However, it was unclear if there exists a strategy that beats quantization-based strategies also by an arbitrarily large factor.

# 3 Vector version of Witsenhausen's counterexample



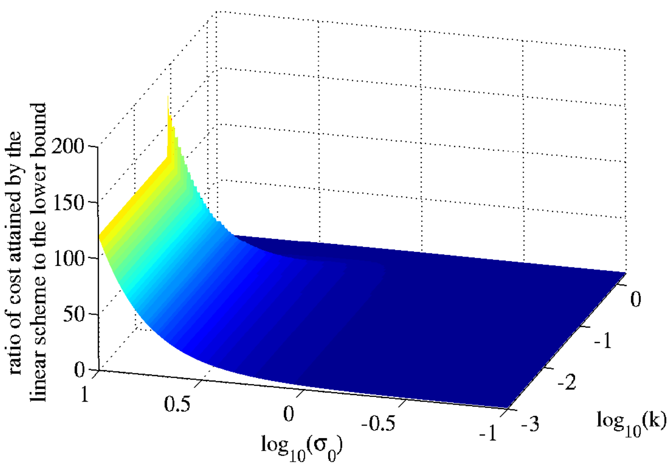

The counterexample can be extended to a vector problem of vector length $m$ [Ho, Kastner and Wong '78]. The asymptotically infinite length extension offers simplification because it allows us to avoid the complications associated with the geometry of finite-dimensional spaces.

### 3.1 Vector quantization (VQ)




Provided the number of quantization points is sufficiently small, they can be decoded correctly at the second controller. Asymptotic cost is $k^2\sigma_w^2$ and 0 for the first and the second stage respectively.

### 3.2 Our lower bound

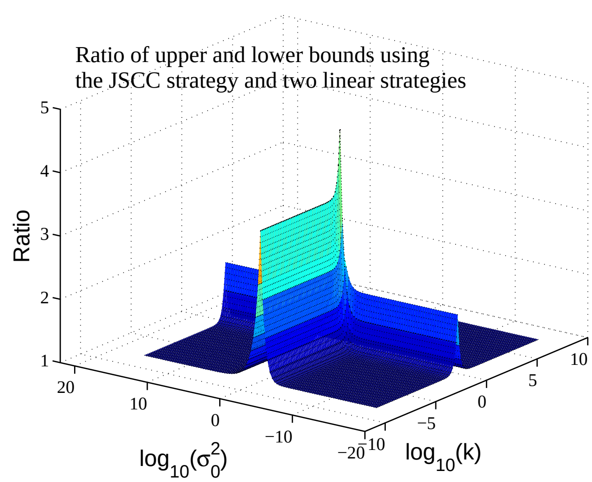



Witsenhausen's lower bounding technique does not extend to the vector case (and further, is loose for the scalar case). We provide the following new lower bound to the optimal costs which is valid for all vector lengths $m \geq 1$,

$$\bar{\mathcal{C}}_{\min} \geq \inf_{P\geq 0} k^2 P + \left( (\sqrt{\kappa(P)} - \sqrt{P})^+ \right)^2,$$

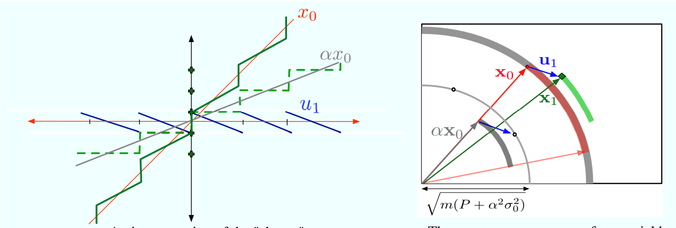
where $\kappa(P) = \frac{\sigma_0^2\sigma_w^2}{\sigma_0^2+2\sigma_0\sqrt{P}+P+\sigma_w^2}$, where $\sigma_w^2 = 1$ is the observation noise variance. This bound is tighter than Witsenhausen's bound for the scalar problem in some cases.

The ratio of optimal linear costs to the lower bound increases to infinity in the small-$k$ large-$\sigma$ regime.



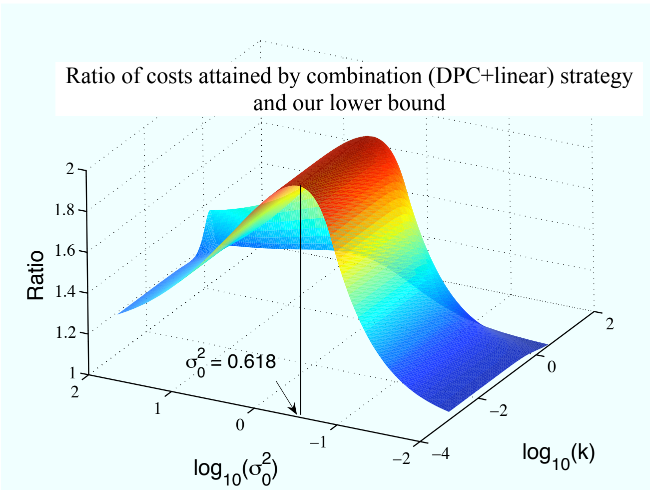

The ratio of our upper bound (obtained by using the vector quantization strategy and two linear schemes of zero-input and zero-forcing) and our lower bound is bounded by 4.45 for the VQ strategy. Analytically, we can show that the ratio is bounded by 11. Observe that the ratio is quite close to 1 for "most" values of $k$ and $\sigma_0^2$.

## 3.3 Improved ratio using Dirty-Paper Coding (DPC) based strategies




An interpretation of the "slopey" quantization strategies of [Baglietto et al]

The exact same sequence of steps yields the DPC-based strategy!

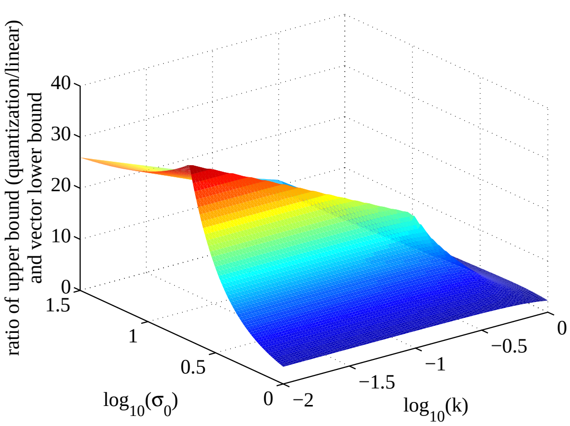



Our second strategy is based on Dirty-Paper Coding (DPC) in information theory *where the shadow state $\alpha\mathbf{x}_0$ is driven to the nearest quantization point*. The strategy turns out to be a vector extension of the "slopey"-quantization strategies in [Baglietto, Parisini and Zoppoli][Lee, Lau and Ho].

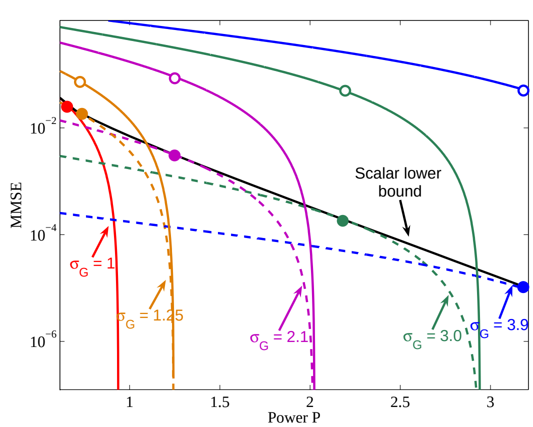
A combination strategy that *divides its power between a linear strategy and the DPC strategy* attains within a factor of 2 for all parameter choices of the infinite length vector Witsenhausen problem.

# 4 The finite-dimensional problem

## 4.1 The scalar case

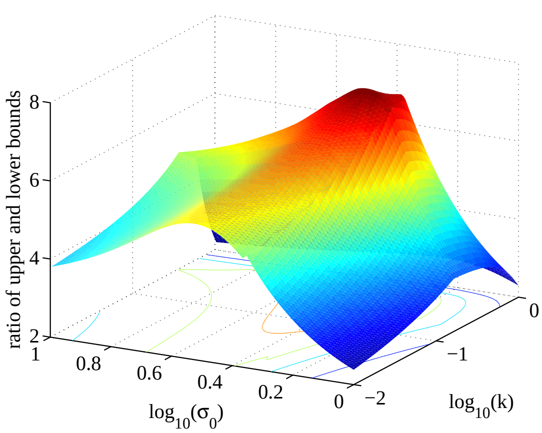



For the scalar case, the ratio of the cost attained by the quantization-based strategies and the lower bound runs off to infinity in the low $k$ regime. Thus, either the quantization-based strategies are bad, or the vector lower bound is not sufficiently tight.



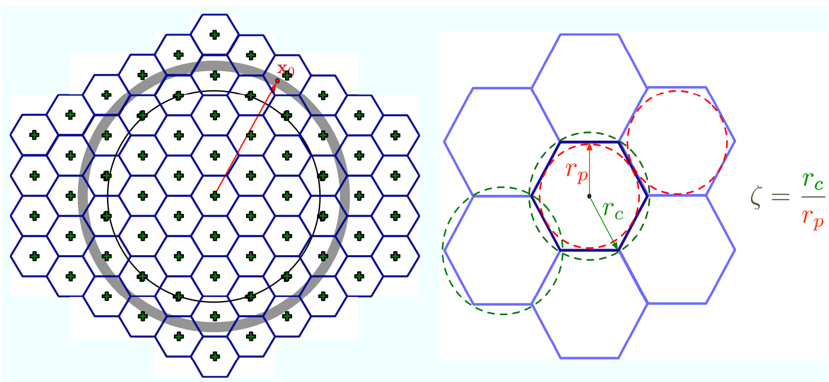

A "Platonic" approach that treats the finite-dimensional world as shadows of the infinite-dimensional world allows us to tighten bounds for finite dimensions. The observation noise can behave as if its variance is much larger, thereby increasing the lower bound. For finite dimensions, there is a non-zero probability (that decays exponentially with the number of dimensions) that this atypical behavior happens.

This style of obtaining "sphere-packing" bounds follows the work of [Blahut '72] and the ensuing extensions to delay [Sahai '06] and neighborhood sizes with bit-error probability [Sahai, Grover '07].

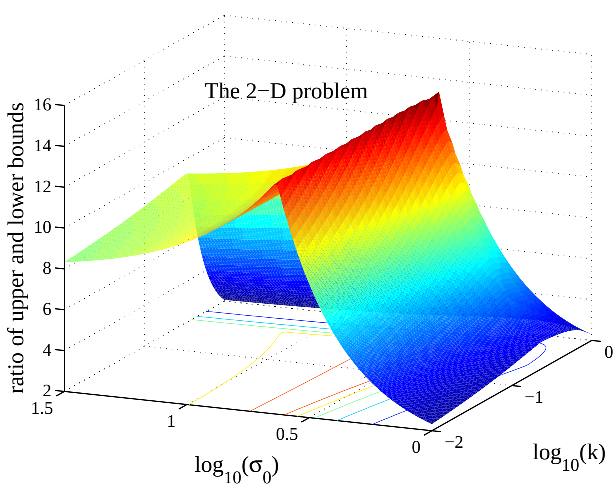



It turns out that the ratio of costs achieved by the optimal combination of linear+ quantization strategies and that of the new bound is bounded by 8 uniformly over all $k$ and $\sigma_0$ for the scalar case.

### 4.2 More than one dimensional counterexample




Since the initial state can fall outside the shell of typical realizations with non-zero probability, we need to tile the space with quantization points. Lattices provide a natural set of such quantization points. Lattices that are good for quantization as well as error resilience have a small $\zeta = \frac{r_c}{r_p}$, where $r_c$ is the covering radius and $r_p$ is the packing radius of the lattice.




For the two-dimensional Witsenhausen counterexample, the hexagonal lattice (of $\zeta = \frac{2}{\sqrt{3}} \approx 1.155$) is better than the square lattice (extension of scalar-quantization, $\zeta = \sqrt{2} \approx 1.414$). The hexagonal lattice attains within a ratio of 15 of the optimal cost. The lower bound is obtained by applying the “Platonic” approach to 2-dimensions.

More generally, we show that lattices can attain within a factor of $300\zeta^2$ of the optimal costs for any finite dimension. Since $\zeta \leq 4$ for all dimensions, lattices attain within a constant factor uniformly over all $k$, $\sigma_0$, and the number of dimensions. The constant can be improved upon by a more careful analysis.

## 5 Summary

This talk intends to bring out the following ideas:

- Witsenhausen's counterexample can be simplified by considering a vector extension. In the limit of long vector lengths, the optimal costs are characterized to within a factor of 2 for all values of $k$ and $\sigma_0^2$. Further, the factor is close to 1 for most values in the $(k, \sigma_0^2)$ parameter space.
- Lattices attain within a constant factor of the optimal costs for the vector Witsenhausen counterexample uniformly over the number of dimensions $m$, $k$, and $\sigma_0$.
- A possible recipe for attacking some distributed control problems is thus as follows:
  - Formulate an infinite length version of the problem, and solve it (perhaps only approximately) using information-theoretic tools.
  - Use sphere-packing and lattice techniques to obtain finite dimensional results.