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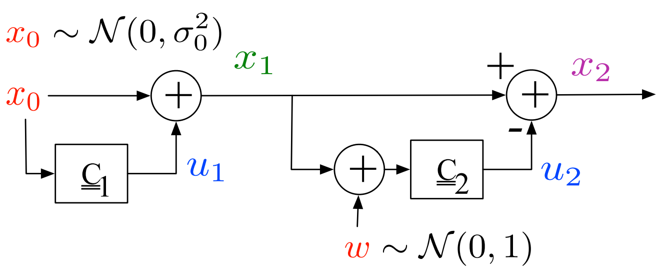
"A vector version of Witsenhausen's counterexample:
Towards Convergence of Control, Communication and Computation"

Conference on Decision and Control (CDC), Dec. 9, 2008.

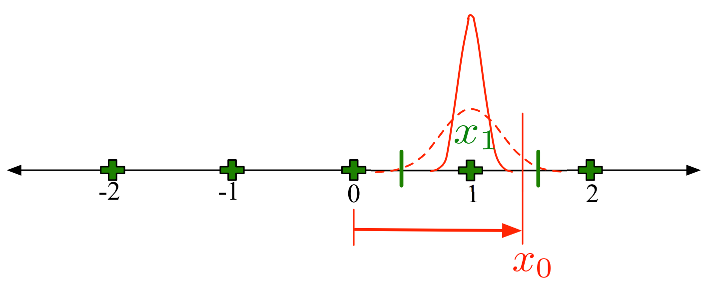
Slides available: www.eecs.berkeley.edu/∼pulkit/CDC08Slides.pdf
This Handout: . . . /∼pulkit/CDC08Handout.pdf
Further discussion and references can be found in the paper.

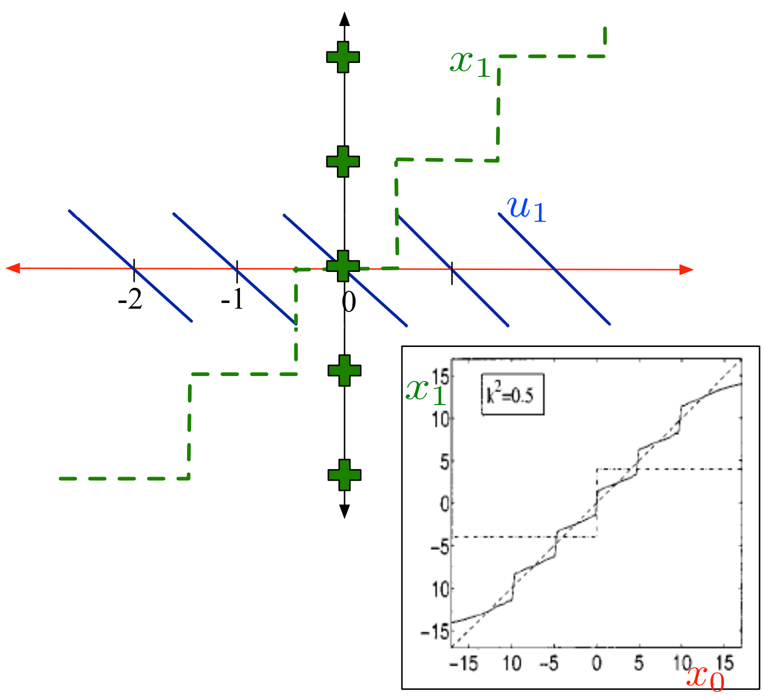
1 Witsenhausen's counterexample



In the first stage, $\underline{\underline{C}}_1$,the first controller, acts on the initial state x_0 using its input u_1 and forces it to x_1. In the second stage, $\underline{\underline{C}}_2$,the second controller observes x_1+w and acts on x_1 to obtain state x_2. The first stage cost is $k^2{u_1}^2$ and the second stage cost is ${x_2}^2$. Thus, the total cost is $\mathcal{C} = k^2{u_1}^2+{x_2}^2$.

2 Quantization based signaling strategies

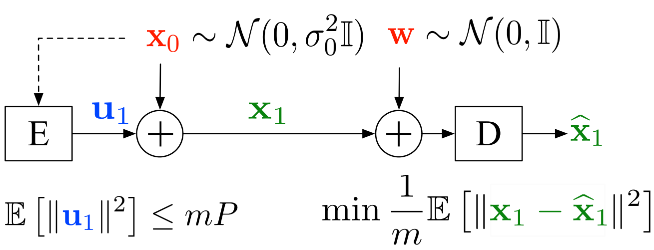




The counterexample contains an implicit communication channel. Using quantization-based signaling strategies, it was shown in [Mitter and Sahai, 99] that the ratio of cost attained by the optimal linear strategy to that attained by the optimal nonlinear strategy can be arbitrarily large.

Numerical optimization results in [Baglietto, Parisini and Zoppoli][Lee, Lau and Ho] suggest that in an interesting regime of small k and large σ_0^2, *soft-quantization based strategies* might be optimal. The inset figure is taken from [Baglietto, Parisini and Zoppoli]

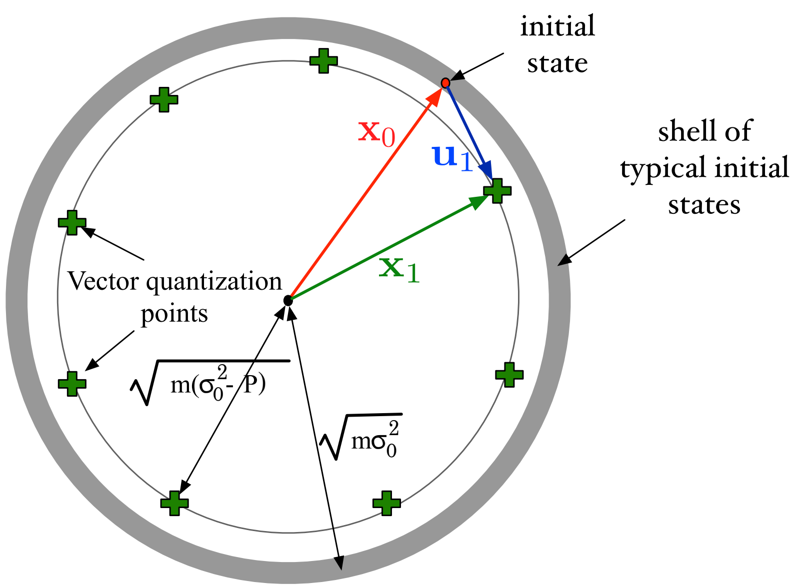
3 Vector version of Witsenhausen's counterexample



We extend the counterexample to a vector problem of vector length m. Interpreted as an information theory problem, the first controller is an "encoder" that drives $\mathbf{x}_0$ to $\mathbf{x}_1$ using an average power constrained input $\mathbf{u}_1$. The second controller, acting as a "decoder", estimates $\mathbf{x}_1$. The objective is to minimize the mean-square error in estimation of $\mathbf{x}_1$.

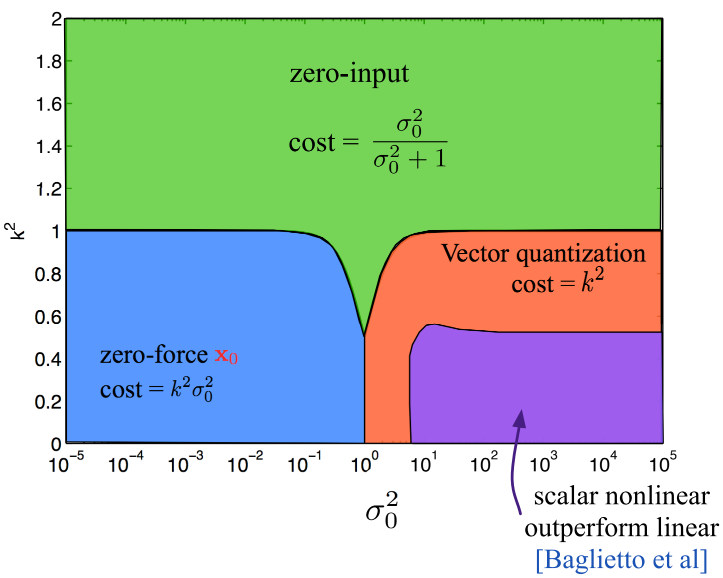
Because of the diagonal state evolution and the diagonal covariance, the best *linear* strategies continue to be scalar.

3.1 Vector quantization strategy



Our first nonlinear strategy is a vector quantization based strategy where *the first controller drives the state* $\mathbf{x}_0$ *to the nearest quantization point*. These quantization points have power smaller than σ_0^2. Provided the number of quantization points is sufficiently small, they can be decoded correctly at the second controller. Asymptotic cost is $k^2\sigma_w^2$ and 0 for the first and the second stage respectively.

When do we need nonlinear strategies?



In the zero-forcing linear strategy, *the first controller chooses* $\mathbf{u} = -\mathbf{x}_0$*, and thus* $\mathbf{x}_1 = 0$. In the zero-input linear strategy, *the first controller uses* $\mathbf{u} = 0$*, and the second controller estimates* $\mathbf{x}_1$ *based on its observation*.

The figure shows which among the three strategies, the two trivial linear schemes and the vector quantization strategy, outperform the other two. Linear strategies perform quite well in situations when σ_0^2 is small, or when k is large. For small k and large σ_0^2, consistent with observations in [Baglietto, Parisini and Zoppoli][Lee, Lau and Ho], the nonlinear vector quantization strategy performs better than the linear strategies.

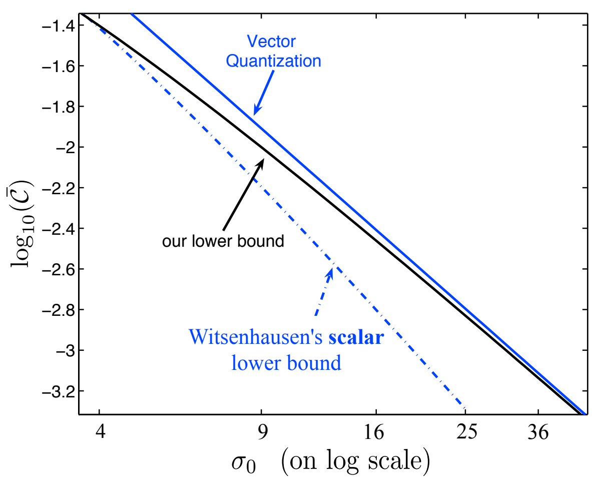
3.2 Lower bounds to the vector Witsenhausen problem

Witsenhausen derived the following lower bound to the total cost for the scalar problem.

$$\bar{\mathcal{C}}_{\min}^{\text{scalar}} \geq \frac{1}{\sigma_0}\int_{-\infty}^{+\infty} \phi\left(\frac{\xi}{\sigma_0}\right) V_k(\xi) d\xi,$$

where $\phi(t) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})$, $V_k(\xi) := \min_a[k^2(a-\xi)^2 + h(a)]$, and $h(a) := \sqrt{2\pi}a^2\phi(a)\int_{-\infty}^{+\infty}\frac{\phi(y)}{\cosh(ay)}dy$.

Our lower bound

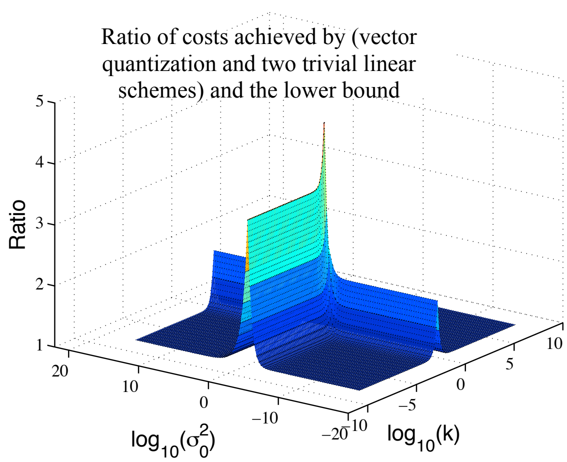


Witsenhausen's lower bounding technique does not extend to the vector case. We provide the following new lower bound to the optimal costs which is valid for all vector lengths $m \geq 1$,

$$\bar{\mathcal{C}}_{\min} \geq \inf_{P\geq 0} k^2P + \left((\sqrt{\kappa(P)} - \sqrt{P})^+\right)^2,$$

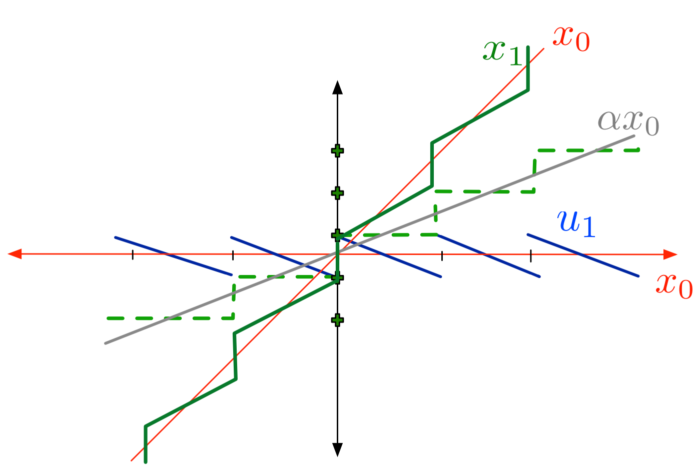
where $\kappa(P) = \frac{\sigma_0^2}{\sigma_0^2 + 2\sigma_0\sqrt{P} + P + 1}$.

For a particular path $k\sigma_0 = 1$ in the parameter space, in the limit of $\sigma_0 \to \infty$, the ratio of our lower bound to that of Witsenhausen diverges to infinity. The vector quantization strategy comes close to our lower bound.



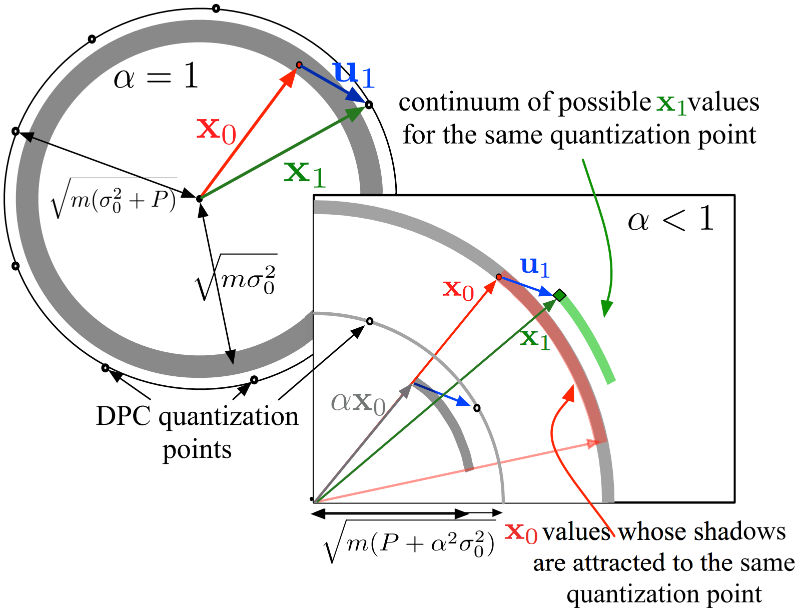
The ratio of our upper bound (obtained by using the vector quantization strategy and the two trivial linear schemes) and our lower bound is no greater than 4.45. Analytically, we can show that the ratio is bounded by 11. Observe that the ratio is quite close to 1 for "most" values of k and σ_0^2.

3.3 Another look at "quantization" to intervals



The quantization to intervals strategies in [Baglietto et al][Lee, Lau, Ho] can be thought of as follows. First, the state x_0 is scaled down to a shadow state αx_0. This shadow state is now quantized, and the resulting u_1 is applied to x_0.

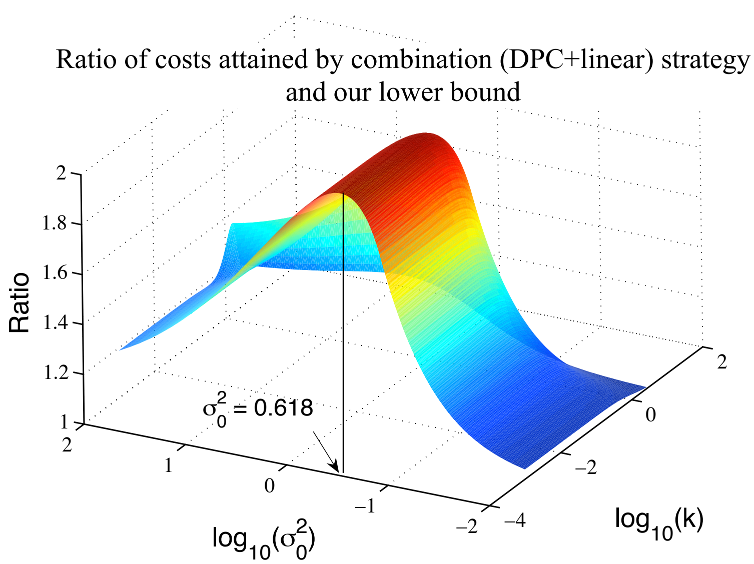
3.4 Dirty-Paper Coding based strategies



In the view of the above interpretation, we propose a second strategy Dirty-Paper Coding (DPC) in information theory *where the shadow state $\alpha\mathbf{x}_0$ is driven to the nearest quantization point*. This is a natural generalization of strategies in [Baglietto et al][Lee, Lau, Ho].

For $\alpha = 1$, the strategy is a hard-quantization strategy which outperforms vector quantization. For $\alpha < 1$, the strategy is conceptually a vector extension of the soft-quantization strategies in [Baglietto, Parisini and Zoppoli][Lee, Lau and Ho]. The first stage cost can be lowered at the expense of nonzero second stage costs.

3.5 Approximately optimal solution



A combination strategy is also proposed. This strategy *divides its power between a linear strategy and the DPC strategy*. It performs at least as well, and in some cases strictly better than the DPC strategy alone.

The figure shows the ratio of the asymptotic cost attained by the combination strategy and our lower bound. This ratio is uniformly bounded by 2 for all values of k and σ_0^2.

4 Summary

This talk intends to bring out the following ideas:

- Witsenhausen's counterexample can be simplified by considering a vector extension. This extension retains the essence of the original counterexample.
- For this extension, in the limit of long vector lengths, the optimal costs are characterized to within a factor of 2 for all values of k and σ_0^2. Further, the factor is close to 1 for most values in the (k, σ_0^2) parameter space.