On the Robustness, Connectivity and Giant Component Size of Random K-out Graphs

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***Abstract*—Random K-out graphs are garnering interest in designing distributed systems including secure sensor networks, anonymous crypto-currency networks, and differentially-private decentralized learning. In these security-critical applications, it is important to model and analyze the resilience of the network to node failures and adversarial captures. Motivated by this, we analyze how the connectivity properties of random K-out graphs vary with the network parameters K, the number of nodes (n), and the number of nodes that get failed or compromised (γ_n). In particular, we study the conditions for achieving *connectivity* with high probability and for the existence of a *giant component* with formal guarantees on the size of the largest connected component in terms of the parameters $n,\ K$, and γ_n. Next, we analyze the property of r*-robustness* which is a stronger property than connectivity and leads to resilient consensus in the presence of malicious nodes. We derive conditions on K and n under which the random K-out graph achieves r-robustness with high probability. We also provide extensive numerical simulations and compare our results on random K-out graphs with known results on Erdős-Rényi (ER) graphs.**

***Index Terms*—Connectivity, giant component, robustness, r-robustness, random graphs, random K-out graphs, security, privacy**

I. Introduction

A. Motivation and Background

In recent years, the rapid proliferation of affordable sensing and computing devices has led to rapid growth in technologies powered by the IoT (Internet of Things). A key challenge in this space is to develop network models for generating a securely connected ad-hoc network in a distributed fashion while minimizing operational costs.

With its unique connectivity properties, a class of random graph models known as the *random K-out graphs* has found many applications in the design of ad-hoc networks. A random K-out graph [3]–[5], denoted as $\mathbb{H}(n; K)$, is an undirected graph with n nodes where each node forms an edge with K distinct nodes chosen uniformly at random. Random K-out graphs are known to achieve connectivity easily, i.e., with far fewer edges $(O(n))$ as compared to classical random graph models including Erdős-Rényi (ER) graphs [3], [6], random geometric graphs [7], and random key graphs [8], which all require $O(n \log n)$ edges for connectivity. In particular, it is known [4], [5] that random K-out graphs are connected with high probability (*whp*) when $K \geq 2$. This had led to their deployment in several applications including random key predistribution schemes for secure communication in sensor networks [9]–[11], differentially-private distributed averaging algorithms [12], anonymity preserving cryptocurrency networks [13], and distributed secure mapping of network addresses in next-generation internet architectures [14].

In the context of sensor networks, random K-out graphs have been used [5], [11], [15] to analyze the performance of the random *pairwise* key predistribution scheme and its heterogeneous variants [16], [17]. The random *pairwise* scheme works as follows. Before deployment, each sensor chooses K others uniformly at random. A unique *pairwise* key is given to each node pair where at least one of them selects the other. After deployment, two sensors can securely communicate if they share a pairwise key. The topology of the sensor network can thus be represented by a random K-out graph; each edge of the random K-out represents a secure communication link between two sensors. Consequently, random K-out graphs have been analyzed to answer key questions on the values of the parameters n and K needed to achieve certain desired properties, including connectivity at the time of deployment [4], [5], connectivity under *link* removals [11], [15], and unassailability [18]. Despite many prior works on random K-out graphs, very little is known about its connectivity properties when some of its *nodes* are removed. This is an increasingly relevant problem since many IoT networks are deployed in remote and hostile environments where nodes may be captured by an adversary, or fail due to harsh conditions.

Another application of random K-out graphs is in distributed learning, where a key goal is to perform computations on user data without compromising the privacy of the users. Random K-out graphs have recently been used to construct the communication graph in a differentially-private federated averaging scheme called the GOPA (GOssip Noise for Private Averaging) protocol [12, Algorithm 1]. According to the GOPA protocol, a random K-out graph is constructed on a set of nodes, of which an *unknown* subset is *dishonest*. It was shown in [12, Theorem 3] that the privacy-utility trade-offs achieved by the GOPA protocol are tightly dependent on the subgraph on

Parts of the material was presented at the 2021 IEEE International Symposium on Information Theory (ISIT) [1] and the 2021 IEEE International Conference on Decision and Control (CDC) [2]. This work was supported in part by the Office of Naval Research (ONR) through Grant N00014-21-1-2547, by the CyLab IoT Initiative, and by the National Science Foundation through Grant CCF-1617934.

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honest nodes being *connected*. When the subgraph on honest nodes is not connected, it was shown that the performance of GOPA is tied to the *size* of the connected components of honest nodes. Since dishonest users can be modeled as randomly deleted nodes, analyzing the connectivity and giant component size of random K-out graphs under node deletions is the key in understanding the performance of the GOPA protocol.

B. Properties of Interest for Random K-out Graphs in Distributed Computing Applications

In the context of applications discussed in the previous section, we can identify several key properties of random K-out graphs that need to be well-understood for performance evaluation and efficient design of the underlying systems. We believe that the graph properties discussed here can also be useful in facilitating new applications of random K-out graphs in different fields, akin to our recent work [19] paving the way to new applications by establishing connectivity guarantees in the finite node regime.

A key metric in quantifying the utility of a network is *connectivity* which is defined as the existence of a path of edges between every node pair [20]. Connectivity ensures that all agents in the network can communicate with one another and no node is isolated from the network. In practice, resource constraints limit the number of links that can be established in the network. Thus, a key goal is to design a *resiliently* connected network while keeping the number of links to be established within operational constraints. Depending on the resource constraints and mission requirements of the application at hand, it may suffice to ensure a weaker notion of connectivity or in cases where agents may routinely fail or compromise, we may even need a stronger notion of connectivity.

In resource-constrained environments, preserving connectivity despite node failures may not be feasible and the goal instead might be to ensure that there is a *large enough*, connected sub-network of users, also known as a *giant* component. For example, it may suffice to aggregate the temperature readings of the majority of sensors deployed in a field to get an estimate of the true temperature. Another example is the power grid network where it is essential to ensure supply to the majority of the users in the event of failures.

In addition to ensuring that the network remains *resiliently* connected in the event of node failures, it is often desirable to ensure that *consensus* can be achieved even in the presence of adversarial agents. In [21], it was shown that network connectivity is not sufficient to characterize consensus when nodes use a certain class of local filtering rules. In particular, it was shown that consensus can be reached in graphs that have the property of being sufficiently *robust*. This is formally quantified by the property of *r-robustness*, which was introduced in [21]. A graph is said to be *r-robust* if, for every disjoint subset pair that partitions the graph, at least one node in one of these subsets is adjacent to at least r nodes in the other set.

The r-robustness property is especially useful in applications of *consensus dynamics*, where parameters of several agents get aligned after a sufficiently long period of local interactions. In another example, it was shown [22] that if the network is $(2F+1)$-robust (for some non-negative integer F), then the nodes in the network can reach consensus even when there are up to F malicious nodes in the neighborhood of every correctly-behaving node. Thus, r-robustness is particularly important for applications based on consensus dynamics in adversarial environments. Moreover, r-robustness is known [22] to be a stronger property than r-connectivity and thus can provide guarantees on the connectivity of the graph when up to $r-1$ nodes in the graph are removed. In the random graph literature, r-robustness has been studied for the ER graph and the Barabási-Albert model in [23], but to the best of our knowledge, there is no prior work on the r-robustness of random K-out graphs except our recent work [2].

C. Main Contributions

With these motivations in mind, this paper aims to fill the gaps in the literature on the connectivity and robustness properties of random K-out graphs. We provide a comprehensive set of results on the connectivity and size of the giant component of the random K-out graph when some of its nodes are *dishonest*, have *failed*, or have been *captured*. We further analyze the conditions required for ensuring r-robustness of the random K-out graph. Our main contributions are summarized below:

1) Let $\mathbb{H}(n; K_n, \gamma_n)$ denote the random graph obtained after removing γ_n nodes, selected uniformly at random, from the random K-out graph $\mathbb{H}(n; K_n)$. We provide a set of conditions for K_n, n, and γ_n under which $\mathbb{H}(n; K_n, \gamma_n)$ is connected *with high probability* (whp). This is done for both cases where $\gamma_n = \Omega(n)$ and $\gamma_n = o(n)$, respectively. Our result for $\gamma_n = \Omega(n)$ (see Theorem 3.1) significantly improves a prior result [24] on the same problem and leads to a *sharp* zero-one law for the connectivity of the random K-out graph under node deletions. Our result for the case $\gamma_n = o(n)$ (see Theorem 3.2) expands the existing threshold of $K_n \geq 2$ required for connectivity by showing that the graph is connected whp for $K_n \geq 2$ even when $o(\sqrt{n})$ nodes are deleted.
2) We derive conditions on K_n, n, γ_n that lead to a *giant component* in $\mathbb{H}(n; K_n, \gamma_n)$ whp and provide an upper bound on the number of nodes not contained in the giant component. This is also done for both cases $\gamma_n = \Omega(n)$ and $\gamma_n = o(n)$; see Theorem 3.3 and Theorem 3.4, respectively. An important consequence of this result is to establish $K_n \geq 2$ as a sufficient condition to ensure *w*hp the existence of a giant component in the random K-out graph despite the removal of $o(n)$ nodes in the network.
3) Using a novel proof technique, we show that $K \geq 2r$ is sufficient to ensure that the random K-out graph $\mathbb{H}(n; K)$ is r-robust *w*hp (see Theorem 3.5). Since it is already known that $\mathbb{H}(n; K)$ is *not* r-robust *w*hp when $K < r$, this result is tight up to at most a multiplicative factor of two (and it is is much tighter than the condition established in [2]).

4) Combining our theoretical results with numerical simulations, we also provide a comparison of random K-out graphs with ER graphs. We determine that random K-out graphs are much more robust in terms of r-robustness property, and also attain connectivity and admit a giant component with fewer edges compared to ER graphs. Our results highlight the usefulness of random K-out graphs as a topology design tool for efficient design of secure, resilient and robust distributed networks.

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D. Organization of the Paper

The rest of this article is organized as follows. In Section II, we introduce the notation used across this article and the network model, the random K-out graph, and extend this model to account for node deletions. In Section III, we present the main results along with the simulation results and provide a detailed discussion. In Section IV, we provide the proof of all Theorems presented in Section III. Conclusions are provided in Section V.

II. Notations and Definitions

All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and probabilistic statements are given with respect to the probability measure $\mathbb{P}$. The complement of an event A is denoted by A^c. The cardinality of a discrete set A is denoted by $|A|$. The intersection of events A and B is denoted by $A \cap B$. We refer to any mapping $K : \mathbb{N}_0 \to \mathbb{N}_0$ as a *scaling* if it satisfies the condition $2 \le K_n < n, \quad n = 2, 3, \ldots$. All limits are understood with n going to infinity. If the probability of an event tends to one as $n \to \infty$, we say that it occurs with high probability (whp). The statements $a_n = o(b_n)$, $a_n = \omega(b_n)$, $a_n = O(b_n)$, $a_n = \Theta(b_n)$, and $a_n = \Omega(b_n)$, used when comparing the asymptotic behavior of sequences $\{a_n\}, \{b_n\}$, have their meaning in the standard Landau notation. The asymptotic equivalence $a_n \sim b_n$ is used to denote the fact that $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. Finally, we let $\langle d \rangle$ denote the mean node degree of a graph.

Definition 2.1 (Random K-out Graph): *The random K-out graph is defined on the vertex set $V := \{v_1, \ldots, v_n\}$ as follows. Let $\mathcal{N} := \{1, 2, \ldots, n\}$ denote the set vertex labels. For each $i \in \mathcal{N}$, let $\Gamma_{n;i} \subseteq \mathcal{N} \setminus i$ denote the set of K_n labels, selected uniformly at random, corresponding to the nodes selected by v_i. It is assumed that $\Gamma_{n;1}, \ldots, \Gamma_{n;n}$ are mutually independent. Distinct nodes v_i and v_j are adjacent, denoted by $v_i \sim v_j$ if at least one of them picks the other. Namely,*

$$v_i \sim v_j \quad \text{if} \quad [j \in \Gamma_{n;i}] \ \vee \ [i \in \Gamma_{n;j}]. \tag{1}$$

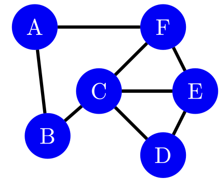

Fig. 1. *An example for a 1-robust graph. We see that with the subset pair $S = \{v_A, v_B\}$ and $S^c = \{v_C, v_D, v_E, v_F\}$, both v_A and v_B in S have only one neighbor in S^c, while v_C and v_F in S^c have only one neighbor in S, meaning both S and S^c are 1-reachable (but not 2-reachable). Further, all other subset pairs that partition the graph are also 1-reachable, leading to the graph in Fig. 1 being 1-robust.*

The set of neighbors of node i is denoted by $\mathcal{V}_i := \{j \in \mathcal{N} \setminus i : v_i \sim v_j\}$, and the degree of node i is denoted as $d_i = |\mathcal{V}_i|$. The random graph defined on the vertex set V through the adjacency condition (1) is called a random K-out graph [3], [5], [25] and is denoted by $\mathbb{H}(n; K_n)$.

Definition 2.2 (Cut): [26, Definition 6.3] For a graph $\mathcal{G}$ defined on the node set $\mathcal{N}$, a *cut* is a non-empty subset $S \subset \mathcal{N}$ of nodes *isolated* from the rest of the graph. Namely, $S \subset \mathcal{N}$ is a cut if there is no edge between S and $S^c = \mathcal{N} \setminus S$. If S is a cut, then so is S^c.

Definition 2.3 (Connected Components): *A pair of nodes in a graph $\mathbb{G}$ are said to be* connected *if there exists a path of edges connecting them. A* connected component *C_i of $\mathbb{G}$ is a subgraph in which any two vertices are connected to each other, and no vertex is connected to a node outside of C_i.*

Definition 2.4 (Giant Component): *For a graph $\mathbb{G}$ with n nodes, a* giant *component exists if its* largest *connected component has size $\Omega(n)$. In that case, the largest connected component is referred to as the giant component of the graph.*

Definition 2.5 (Connectivity): A graph $\mathbb{G}$ is *connected* if there exists a path of edges between every pair of its vertices.

Definition 2.6 (r-connectivity): A graph is r-connected if it remains connected after the removal of *any* set of $r-1$ (or, fewer) nodes or edges.

Definition 2.7 (r-reachable Set): [21, Definition 6] *For a graph $\mathbb{G}$ and a subset S of nodes $S \subset \mathcal{N}$, we say S is r-reachable if $\exists i \in S : |\mathcal{V}_i \setminus S| \geq r$, where $r \in \mathbb{Z}^+$. In other words, S is an r-reachable set if it contains a node that has at least r neighbors outside S.*

Definition 2.8 (r-robust Graph): [21, Definition 6] *A graph $\mathbb{G}$ is r-robust if for every pair of nonempty, disjoint subsets of $\mathcal{N}$ that partition $\mathcal{N}$, at least one of these subset pairs is r-reachable, where $r \in \mathbb{Z}^+$.*

It was shown in [22] that if a graph is r-robust, it is at least r-connected. Thus, r-robustness is a stronger property than r-connectivity. It is also easy to see that when $r = 1$, the properties of r-robustness and r-connectivity are equivalent.

A main goal of this paper is to study the connectivity and giant component size of random K-out graphs when some of its nodes are *failed*, *captured*, or *dishonest*. To this end, we consider the following model of random K-out graphs under random removal of nodes. We first let γ_n denote the number of removed nodes and assume, for simplicity, that they are selected uniformly at random among all nodes in V. Further, we let $D \subset V$, $|D| = \gamma_n$ denote the set of deleted nodes. We then define $\mathbb{H}(n; K_n, \gamma_n)$ on the vertex set $R = V \setminus D$ and the corresponding set of labels $\mathcal{N}_R$, such that distinct vertices v_i and v_j (both in R) are adjacent if they were adjacent in $\mathbb{H}(n; K_n)$; i.e., if $[j \in \Gamma_{n;i}] \vee [i \in \Gamma_{n;j}]$. For each $i \in \mathcal{N}_R$, the set of labels adjacent to node v_i in $\mathbb{H}(n; K_n, \gamma_n)$ is denoted by $\Gamma_{n, \gamma_n; i} \subseteq \mathcal{N}_R \setminus i$.

III. Main Results

Our main results are presented in Theorems 3.1- 3.5 below. Each Theorem addresses a design question as to how the parameter K_n should be chosen to satisfy the desired property on robustness, connectivity or the size of the giant component; see Table I for a summary of the main results. The results on connectivity and the size of the giant component are for $\mathbb{H}(n; K_n, \gamma_n)$, i.e., the random K-out graph when γ_n nodes are deleted, while the result on r-robustness is given for the original graph $\mathbb{H}(n; K_n)$ (without any node deletion). We provide the proofs of all results in Section IV.

A. Results on Connectivity

Let $P(n, K_n, \gamma_n) = \mathbb{P}[\mathbb{H}(n; K_n, \gamma_n) \text{ is connected}]$.

Theorem 3.1: *Let $\gamma_n = \alpha n$ with α in $(0, 1)$, and consider a scaling $K : \mathbb{N}_0 \to \mathbb{N}_0$ such that with $c > 0$ we have*

$$K_n \sim c \cdot r_1(\alpha, n), \text{ where } \quad r_1(\alpha, n) = \frac{\log n}{1 - \alpha - \log \alpha} \tag{2}$$

is the threshold function. Then, we have

$$\lim_{n \to \infty} P(n, K_n, \gamma_n) = \begin{cases} 1, & \text{if } \quad c > 1 \\ 0, & \text{if } \quad 0 < c < 1. \end{cases} \tag{3}$$

The proof of the *one-law* in (3), i.e., that $\lim_{n \to \infty} P(n, K_n, \gamma_n) = 1$ if $c > 1$, is given in Section IV. The *zero-law* of (3), i.e., that $\lim_{n \to \infty} P(n, K_n, \gamma_n) = 0$ if $c < 1$, was established previously in [24, Corollary 3.3]. There, a one-law was also provided: under (2), it was shown that $\lim_{n \to \infty} P(n, K_n, \gamma_n)$ if $c > \frac{1}{1-\alpha}$, leaving a gap between the thresholds of the zero-law and the one-law. Theorem 3.1 presented here fills this gap by establishing a tighter one-law, and constitutes a *sharp* zero-one law; e.g., when $\alpha = 0.5$, the one-law in [24] is given with $c > 2$, while we show that it suffices to have $c > 1$.

Theorem 3.2: *Consider a scaling $K : \mathbb{N}_0 \to \mathbb{N}_0$.*
a) If $\gamma_n = o(\sqrt{n})$, then we have

$$\lim_{n \to \infty} P(n, K_n, \gamma_n) = 1, \quad \text{if} \quad K_n \geq 2 \ \forall n \tag{4}$$

b) If $\gamma_n = \Omega(\sqrt{n})$ and $\gamma_n = o(n)$, and if for some sequence w_n, it holds that

$$K_n = r_2(\gamma_n) + \omega_n, \text{ where } \quad r_2(\gamma_n) = \frac{\log(\gamma_n)}{\log 2 + 1/2}$$

is the threshold function, then we have

$$\lim_{n \to \infty} P(n, K_n, \gamma_n) = 1, \quad \text{if} \quad \lim_{n \to \infty} \omega_n = \infty \tag{5}$$

We remind that random K-out graph is known [4], [5] to be connected whp when $K_n \geq 2$. Part (a) of Theorem 3.2 extends this result by showing that having $K_n \geq 2$ is sufficient for the random K-out graph to remain connected whp even when $o(\sqrt{n})$ of its nodes (selected randomly) are deleted. We believe that this result will further facilitate the application of random K-out graphs in a wide range of applications where connectivity despite node failures is crucial.

B. Results on the Size of the Giant Component

Let $C_{max}(n, K_n, \gamma_n)$ denote the set of nodes in the *largest* connected component of $\mathbb{H}(n; K_n, \gamma_n)$ and let $P_G(n, K_n, \gamma_n, \lambda_n) := \mathbb{P}[|C_{max}(n, K_n, \gamma_n)| > n - \gamma_n - \lambda_n]$. Namely, $P_G(n, K_n, \gamma_n, \lambda_n)$ is the probability that less than λ_n nodes are *outside* the largest component of $\mathbb{H}(n; K_n, \gamma_n)$.

Theorem 3.3: *Let $\gamma_n = o(n)$, $\lambda_n = \Omega(\sqrt{n})$ and $\lambda_n \leq \lfloor (n - \gamma_n)/3 \rfloor$. Consider a scaling $K : \mathbb{N}_0 \to \mathbb{N}_0$ and let*

$$r_3(\gamma_n, \lambda_n) = 1 + \frac{\log(1 + \gamma_n/\lambda_n)}{\log 2 + 1/2}$$

be the threshold function. Then, we have

$$\lim_{n \to \infty} P_G(n, K_n, \gamma_n, \lambda_n) = 1, \quad \text{if} \quad K_n > r_3(\gamma_n, \lambda_n), \ \forall n.$$

We remark that if $\lambda_n = \beta n$ with $0 < \beta < 1/3$, then $r_3(\gamma_n, \lambda_n) = 1 + o(1)$. This shows that when $\gamma_n = o(n)$, it suffices to have $K_n \geq 2$ for $\mathbb{H}(n; K_n, \gamma_n)$ to have a giant component containing $(1 - \beta)n$ nodes for arbitrary $0 < \beta < 1/3$. Put differently, by choosing $K_n \geq 2$, we ensure that even when $\gamma_n = o(n)$ nodes are removed, the rest of the network contains a connected component whose *fractional* size is arbitrarily close to 1.

Theorem 3.4: *Let $\gamma_n = \alpha n$ with α in $(0, 1)$, and $\lambda_n \leq \lfloor \frac{(1-\alpha)n}{3} \rfloor$. Consider a scaling $K : \mathbb{N}_0 \to \mathbb{N}_0$ and let*

$$r_4(\alpha, \lambda_n) = 1 + \frac{\log(1 + \frac{\alpha n}{\lambda_n}) + \alpha + \log(1 - \alpha)}{\frac{1-\alpha}{2} - \log \frac{1+\alpha}{2}}$$

be the threshold function. Then, we have

$$\lim_{n \to \infty} P_G(n, K_n, \alpha, \lambda) = 1, \quad \text{if} \quad K_n > r_4(\alpha, x_n), \ \forall n.$$

It can be seen from this result that K_n needs to scale as $K_n \sim \log(\frac{\alpha n}{\lambda_n})$ for a random K-out graph to have a giant component of size $n - \lambda_n$ when αn of its nodes are removed (or, if each node is independently removed with probability $0 < \alpha < 1$). We also remark that the threshold $r_4(\alpha, \lambda_n)$ is finite when $\lambda_n = \Omega(n)$. This shows that even when a positive fraction of the nodes of the random K-out graph are removed, a finite K_n is still sufficient to have a giant component of size $\Omega(n)$ in the graph. This result can be useful in applications where it is required to maintain a giant component as efficiently (i.e., with as fewest edges) as possible even when large scale node failures take place.

C. Result on Robustness

Theorem 3.5: *Consider a scaling $K : \mathbb{N}_0 \to \mathbb{N}_0$, for all $r \in \mathbb{Z}^+$, we have*

$$\lim_{n \to \infty} \mathbb{P}[\mathbb{H}(n; K_n) \textit{ is } r\textit{-robust}] = 1, \quad \text{if} \quad K_n \geq 2r$$

Desired Property	Minimum K_n needed to achieve the property	Theorem
Connectivity $\gamma_n = \alpha n$	$(1+\epsilon)\frac{\log n}{1-\alpha-\log\alpha}$	Thm. 3.1
Connectivity $\gamma_n = o(\sqrt{n})$	$K_n \geq 2$	Thm. 3.2(a)
Connectivity $\gamma_n = \omega(\sqrt{n})$, $\gamma_n = o(n)$	$\frac{\log(\gamma_n)}{\log 2+1/2} + \omega(1)$	Thm. 3.2(b)
Giant Component, $\gamma_n = o(n)$, $\lambda_n = o(n)$	$1 + \frac{\log(1+\frac{\gamma_n}{\lambda_n})}{\log(2)+1/2}$	Thm. 3.3
Giant Component, $\gamma_n = \alpha n$, $\lambda_n = o(n)$	$K_n \geq 2$	Thm. 3.3
Giant Component, $\lambda_n < \lfloor\frac{(1-\alpha)n}{3}\rfloor$, $\gamma_n = \alpha n$	$1 + \frac{\log(1+\frac{\gamma_n}{\lambda_n})+\alpha+\log(1-\alpha)}{\frac{1}{2}-\log(\frac{1+\alpha}{2})}$	Thm. 3.4
r-robustness	$K_n \geq 2r$	Thm. 3.5

TABLE I

Summary of our main results providing a condition on K_n needed to achieve a desired property in $\mathbb{H}(n; K_n; \gamma_n)$ (whp) where γ_n denotes the number of deleted nodes. For the giant component, the desired property is defined as its size to be at least $n - \lambda_n$. In the first row, the result is for any $\epsilon > 0$. On the fifth row, we have α in $(0, 1/3)$ and on the sixth row, we have α in $(0, 1)$.

It was previously established in [2] that $\mathbb{H}(n; K_n)$ is r-robust *whp* if $K_n > \frac{2r(\log(r)+\log(\log(r)+1)}{\log(2)+1/2-\log\left(1+\frac{\log(2)+1/2}{2\log(r)+5/2+\log(2)}\right)}$. The threshold on Theorem 3.5 is much smaller, hence constitutes a sharper one-law for r-robustness.

D. Discussion

In Theorem 3.1, we improve the results given in [24], and with this, we close the gap between the zero law and the one law, and hence establish a sharp zero-one law for connectivity when $\gamma_n = \Omega(n)$ nodes are deleted from $\mathbb{H}(n; K_n, \gamma_n)$. In Theorem 3.2, we establish that the graph $\mathbb{H}(n; K_n, \gamma_n)$ with $\gamma_n = o(n)$ is connected whp when $K_n \sim \log(\gamma_n)$; and when $\gamma_n = o(\sqrt{n})$, $K_n \geq 2$ is sufficient for connectivity. The latter result is especially important, since $K_n \geq 2$ is the previously established threshold for connectivity [4]. We improve this result by showing that the graph is still connected with $K_n \geq 2$ even after $o(\sqrt{n})$ nodes (selected randomly) are deleted.

Since most distributed systems require connectivity in the event of node failures, our results can be useful in many applications of distributed systems, particularly when the resources on each node is limited and it is critical to achieve desired connectivity and robustness properties using as few edges as possible. For example, in wireless sensor networks, knowing the minimum conditions needed for connectivity or giant component size under such failures is crucial as it enables designing them with fewest edges possible per node [18], [27], which reduces the communication overhead and potentially the cost of the hardware on each node.

We also note that Theorems 3.3 - 3.4 constitute the first results concerning the giant component size of random K-out graphs under randomly deleted nodes. In particular, these results help choose the value of K_n for any anticipated level of node failure and for any given giant component size required, enabling the designs of distributed systems to compromise between efficiency, robustness, and the required giant component size. Thus, we expect these results to be useful in applications where connectivity is not a stringent condition under node failures, and instead having a certain giant component size is sufficient to continue the operation of the system.

In Theorem 3.5, we establish that a random K-out graph is r-robust *whp* when $K_n \geq 2r$ for any $r \in \mathbb{Z}^+$. This is a much sharper one-law than the previous result given in [2] where it was shown that K_n needs to scale as $K_n \sim r\log(r)$ for r-robustness. This tighter result was made possible through several novel steps introduced here. While the proofs in prior work [2], [28] also rely on finding upper bounds on the probability of having at least one subset that is not r-reachable, they tend to utilize standard upper bounds for the binomial coefficients $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and a union bound to establish them. Instead, our proof uses extensively the Beta function $B(a, b)$ and its properties to obtain tighter upper bounds on such probabilities, which then enables us to establish a much sharper one-law for r-robustness of random K-out graphs. We believe this result will pave the way for further applications of random K-out graphs in distributed computing applications such as the design of consensus networks in the presence of adversaries.

It is also of interest to compare the threshold of r-robustness and r-connectivity in random K-out graphs. For Erdős-Rényi graphs, the threshold for r-connectivity and r-robustness have been shown [28] to coincide with each other. For random K-out graphs, we know from [4] that $\mathbb{H}(n; K_n)$ is r-connected *whp* whenever $K_n \geq r$, and it is *not* r-connected *whp* if $K_n < r$. This leaves a factor of 2 difference between the condition $K_n \geq 2r$ we established for r-robustness here and the threshold of r-connectivity. Put differently, we know from [4] and Theorem 3.5 that for any $r = 2, 3, \ldots$

$$\lim_{n\to\infty} \mathbb{P}\left[\mathbb{H}(n; K_n) \text{ is } r\text{-robust}\right] = \begin{cases} 1, & \text{if} \quad K_n \geq 2r \\ 0, & \text{if} \quad K_n < r. \end{cases} \tag{6}$$

For $r = 1$, it is instead known that $\lim_{n\to\infty} \mathbb{P}\left[\mathbb{H}(n; K_n) \text{ is } r\text{-robust}\right] = 1$ if only if $K_n \geq 2r = 2$. Since the currently established conditions for the zero-law and one-law of r-robustness are not the same for random K-out graphs (unlike ER graphs where the two thresholds coincide), there is a question as to whether our threshold of $2r$ is the tightest possible for r-robustness. This is currently an open problem and would be an interesting direction for future work, e.g., by establishing a tighter zero-law for r-robustness that coincides with the one-law of Theorem 3.5.

To put all these results in perspective, we provide comparisons of our results with the results from an Erdős-Rényi graph $G(n, p)$, which is one of the most commonly used random graph models. Firstly, in terms of r-robustness, it was shown in [28] that ER graph G(n; p) is r-robust *whp* if $p_n = \frac{\log(n)+(r-1)\log(\log(n))+\omega(1)}{n}$, which translates to a mean node degree of $\langle d\rangle \sim \log(n) + (r-1)\log(\log(n))$. Since the mean node degree required for random K-out graphs scales

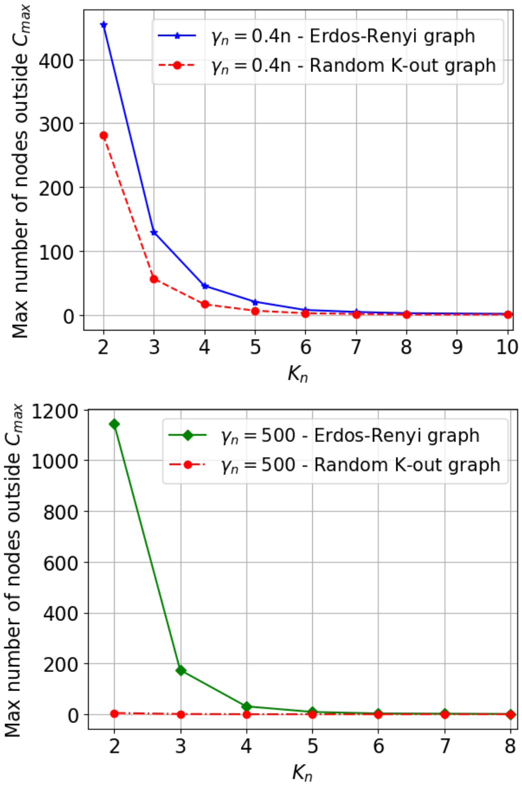

Fig. 2. *Comparison of maximum number of nodes outside the giant component of a random K-out graph* $\mathbb{H}(n; K_n;\ _n)$ *and an ER graph with same mean node degree when* $n = 5000$, $_n = 0{:}4n$ *(Top); and when* $n = 50{;}000$ *and* $_n = 500$ *(Bottom). Each data-point is obtained through 1000 experiments.*

as $\langle d \rangle \sim 2r$, we can conclude that for large n, random K-out graphs can be ensured to be r-robust *whp* at a mean node degree significantly smaller than the mean node degree required for an ER graph. In terms of connectivity, an ER graph becomes connected whp if $p > \log n/n$ [29], and this translates to having a mean node degree of $\langle d \rangle \sim \log n$. Similarly, when $o(\sqrt{n})$ nodes are removed, the mean node degree required for connectivity scales with $\langle d \rangle \sim \log n$ [6]. The $\langle d \rangle$ required for the random K-out graph to be connected whp is much lower, with $\langle d \rangle = O(1)$ when $o(\sqrt{n})$ nodes are removed, and $\langle d \rangle \sim \log(\gamma_n)$ when $\gamma_n = \Omega(\sqrt{n})$ nodes are removed.

Next, we compare the size of the giant component in random K-out graphs and ER graphs under node removals in the finite node regime. We examine the maximum number of nodes outside the giant component out of 1000 experiments of a random K-out graph $\mathbb{H}(n; K_n, \gamma_n)$ and an ER graph $G(n, p)$ with the same mean node degree when γ_n nodes are removed from both graphs. To ensure that both graphs have the same mean node degree, p in the ER graph is selected as $p = 2K_n/n$. The results are given in Fig. 2 for $n = 5000$, $\gamma_n = 0.4n$ on (Top), and $n = 50,000$, $\gamma_n = 500$ on (Bottom). As can be seen, the random K-out graph tends to have fewer nodes outside of the giant component than the ER graph and this difference is more pronounced when γ_n is smaller.

In conclusion, we see that when both graphs have the same mean node degree, random K-out graphs are more robust than ER graphs in terms of connectivity and giant component size under random node removal, and also in terms of the r-robustness property. This reinforces the efficiency of the K-out construction in many distributed computing applications where connectivity in the event of node failures or adversarial capture of nodes is crucial. Similarly, the fact that random K-out graphs tend to achieve r-robustness with fewer edges per node than ER graphs (for any $r = 1, 2, \ldots$), makes it more suitable in applications based on distributed consensus.

E. Simulation Results

Since our results are asymptotic in nature, i.e., they have been established in the limit $n \to \infty$, an important question is whether they can also be useful in practical settings where the number n of nodes is finite. We check the usefulness to validate Theorems 3.1 - 3.4 under practical settings, To answer this, we examine the probability of connectivity and the number of nodes outside the giant component for the graph $\mathbb{H}(n; K_n, \gamma_n)$ (random K-out graph with deleted nodes) through computer simulations in two different setups[1].

In the first setup, we consider the case where the number of deleted nodes, $\gamma_n = \alpha n$, with α in $(0, 1)$. We generate instantiations of the random graph $\mathbb{H}(n; K_n, \gamma_n)$ with $n = 5000$, varying K_n in the interval $[1, 25]$ and consider several α values in the interval $[0.1, 0.8]$. Then, we record the empirical probability of connectivity of the graph $\mathbb{H}(n; K_n, \gamma_n)$ and λ_n from 1000 independent experiments for each (K_n, α) pair. The results of this experiment are shown in Fig. 3 and Fig. 4. Fig. 3 (Top) depicts the empirical probability of connectivity of $\mathbb{H}(n; K_n, \gamma_n)$. The vertical lines stand for the critical threshold of connectivity obtained from Theorem 3.1. In each curve, $P(n, K_n, \gamma_n)$ exhibits a threshold behaviour as K_n increases, and the transition from $P(n, K_n, \gamma_n) = 0$ to $P(n, K_n, \gamma_n) = 1$ takes place around $K_n = \frac{\log n}{1 \quad \log}$, the threshold established in (2), reinforcing the usefulness of Theorem 3.1 under practical settings.

In Fig. 4, the *maximum* number of nodes outside the giant component in 1000 experiments is plotted for each parameter pair. For comparison, we also plot the upper bound on $n - \gamma_n - |C_{max}|$ obtained from Theorem 3.4 by taking the maximum γ_n value that gives a threshold less than or equal to the K_n value tested in the simulation. As can be seen, for any K_n and γ_n value, the experimental maximum number of nodes outside the giant component is smaller than the upper bound obtained from Theorem 3.4, validating the usefulness of this result in the finite node regime.

The goal of the second experimental setup is to examine the case where the number of deleted nodes is $\gamma_n = o(n)$. As before, we generate instantiations of the random graph $\mathbb{H}(n; K_n, \gamma_n)$, with $n = 50,000$, varying K_n in $[2, 5]$, and varying λ_n in $[10, 2000]$. For each (K_n, γ_n) pair, the maximum number of nodes outside the giant component in 1000 experiments is recorded; if no node is outside the giant component, then it is understood that the graph is connected. In Fig. 3 (Bottom), the maximum number of nodes outside the giant component observed in 1000 experiments is depicted as a function of K_n. The plots for $\gamma_n = 10$ and $\gamma_n = 100$ are

[1] Determining whether a graph is r-robust is a co-NP-complete problem [28] making it not feasible to check the usefulness of Theorem 3.5 through computer simulations.

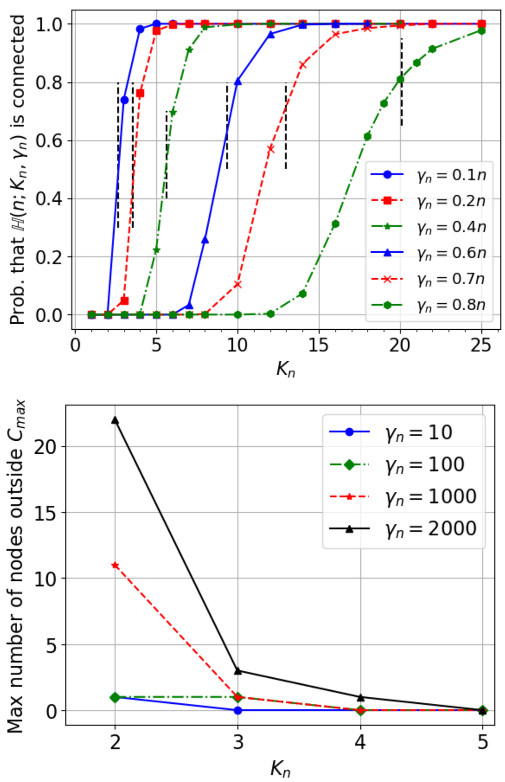

Fig. 3. *(Top) Empirical probability that $\mathbb{H}(n;K_n;\gamma_n)$ is connected for $n = 5000$ calculated from 1000 experiments. The vertical lines are the theoretical thresholds given by Theorem 3.1. (Bottom) Maximum number of nodes outside the giant component of $\mathbb{H}(n;K_n;\gamma_n)$ for $n = 50{,}000$ in 1000 experiments.*

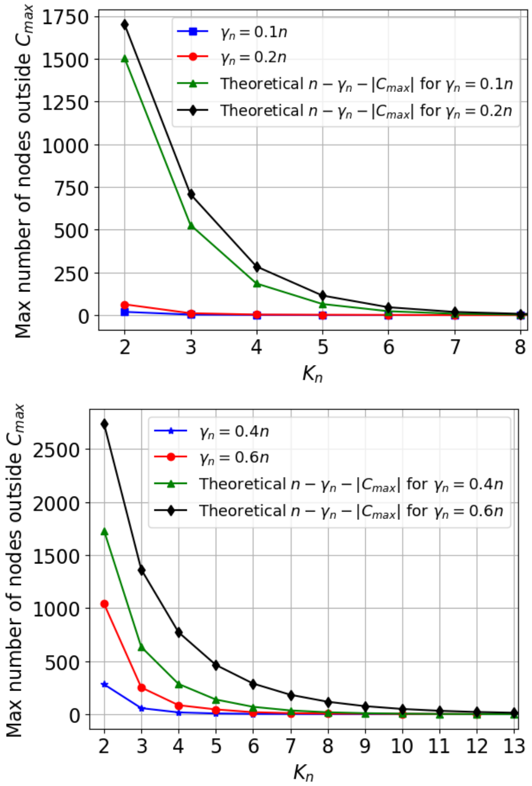

Fig. 4. *Maximum number of nodes outside the giant component of $\mathbb{H}(n;K_n;\gamma_n)$ for $n = 5000$ and $\gamma_n = 0.1n$, $\gamma_n = 0.2n$ cases (Top); and for $n = 5000$ and $\gamma_n = 0.4n$, $\gamma_n = 0.6n$ cases (Bottom), obtained through 1000 experiments along with respective plot of theoretical $n - \gamma_n - |C_{max}|$.*

considered to represent the case $\gamma_n = o(\sqrt{n})$ in Theorem 3.2(a). We see that there is at most one node outside the giant component for $\gamma_n = 10$ and $\gamma_n = 100$, even when $K_n = 2$. This shows that the asymptotic behavior given in Theorem 3.2(a), i.e., that random K-out graph remains connected if $o(\sqrt{n}$ nodes are deleted, already appears when $n = 50,000$. The plots for $\gamma_n = 1000$ and $\gamma_n = 2000$ are used to check the case $\gamma_n = w(\sqrt{n})$ and $\gamma_n = o(n)$ in Theorem 3.2(b). The thresholds on K_n for these γ_n values, obtained using Theorem 3.2(b) are $r_2(1000) = 6.79$ and $r_2(2000) = 7.37$, rounded to two digits after decimal (the $\omega(1)$ term in Theorem 3.2(b) is ignored due to n having a finite value in the simulations). It is clear from the plot that when $K_n \geq 4$, the graph with $\gamma_n = 1000$ is connected, while $K_n \geq 5$ suffices to ensure connectivity when $\gamma_n = 2000$. Thus, selecting K_n above the theoretical thresholds given in 3.2(b) is seen to ensure the connectivity of the graph in the finite node regime as well, supporting the usefulness of Theorem 3.2(b) in practical cases.

Finally, in Fig. 5, the maximum number of nodes outside the giant component in 1000 experiments is plotted as a function of K_n. For comparison, we also plot the upper bound on $n - |C_{max}|$ obtained from Theorem 3.4. In particular, for each Theorem, the maximum γ_n value that gives a threshold less than or equal to the K_n value tested in the simulation is found. Then, the lowest of these maximum γ_n values is used as the theoretical $n - |C_{max}|$ value. As can be seen, for any K_n and γ_n value, the experimental maximum number of nodes outside the giant component is smaller than the upper bounds obtained from Theorem 3.4, reinforcing the usefulness of our results in

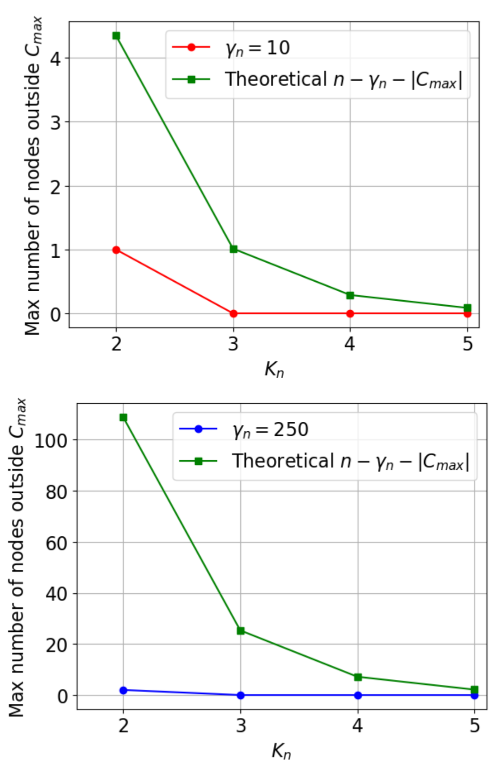

Fig. 5. *Maximum number of nodes outside the giant component of $\mathbb{H}(n;K_n;\gamma_n)$ for $n = 50{,}000$ and $\gamma_n = 10$ cases (Top); and for $n = 50{,}000$ and $\gamma_n = 250$ cases (Bottom), obtained through 1000 experiments along with the plot of theoretical $n - \gamma_n - |C_{max}|$.*

the finite node regime.

IV. Proofs of Main Results

In this section, we provide the proof of all Theorems presented in Section III.

A. Preliminary Steps for Proving Theorems 3.1 - 3.4

Since the preliminary steps related to the proofs of 3.1 - 3.4 are the same, in this Section we present these steps. First, recall from Section II that the metrics connectivity and the size of the giant component under node removals are defined for the graph $\mathbb{H}(n;K_n,\gamma_n)$, where the set D of nodes is removed from the graph $\mathbb{H}(n;K_n)$; also recall that $R = \mathrm{V}\setminus D$. Let $\mathcal{N}_R$ denote the set of labels of the vertex set of $\mathbb{H}(n;K_n,\gamma_n)$ and let $\mathcal{E}_n(K_n,\gamma_n;S)$ denote the event that $S\subset\mathcal{N}_R$ is a cut in $\mathbb{H}(n;K_n,\gamma_n)$ as per Definition 2.2. The event $\mathcal{E}_n(K_n,\gamma_n;S)$ occurs if no nodes in S pick neighbors in S^{c}, and no nodes in S pick neighbors in S^{c}. Note that nodes in S or S^{c} can still pick neighbors in the set $\mathcal{N}_D$. Thus, we have

$$\mathcal{E}_n(K_n,\gamma_n;S) = \bigcap_{i\in S}\bigcap_{j\in S^{\mathrm{c}}}\left(\{i\notin\Gamma_{n,\gamma_n;j}\}\cap\{j\notin\Gamma_{n,\gamma_n;i}\}\right).$$

Let $\mathcal{Z}(\lambda_n;K_n,\gamma_n)$ denote the event that $\mathbb{H}(n;K_n,\gamma_n)$ has no cut $S\subset\mathcal{N}_R$ with size $\lambda_n\le|S|\le n-\gamma_n-\lambda_n$ where $x:\mathbb{N}_0\to\mathbb{N}_0$ is a sequence such that $\lambda_n\le(n-\gamma_n)/2\ \forall n$. In other words, $\mathcal{Z}(\lambda_n;K_n,\gamma_n)$ is the event that there are no cuts in $\mathbb{H}(n;K_n,\gamma_n)$ whose size falls in the range $[\lambda_n, n-\gamma_n-\lambda_n]$.

Lemma 4.1: [17, Lemma 4.3] For any sequence $x:\mathbb{N}_0\to\mathbb{N}_0$ such that $\lambda_n\le\lfloor(n-\gamma_n)/3\rfloor$ for all n, we have

$$\mathcal{Z}(\lambda_n;K_n,\gamma_n)\Rightarrow|C_{max}(n,K_n,\gamma_n)|>n-\gamma_n-\lambda_n. \quad (7)$$

Lemma 4.1 states that if the event $\mathcal{Z}(\lambda_n;K_n,\gamma_n)$ holds, then the size of the largest connected component of $\mathbb{H}(n;K_n,\gamma_n)$ is greater than $n-\gamma_n-\lambda_n$; i.e., there are less than λ_n nodes outside of the giant component of $\mathbb{H}(n;K_n,\gamma_n)$. Also note that $\mathbb{H}(n;K_n,\gamma_n)$ is connected if $\mathcal{Z}(\lambda_n;K_n,\gamma_n)$ takes place with $\lambda_n=1$, since a graph is connected if no node is outside the giant component. In order to establish the Theorems 3.1-3.4., we need to show that $\lim_{n\to\infty}\mathbb{P}[\mathcal{Z}(\lambda_n;K_n,\gamma_n)^{\mathrm{c}}]=0$ with λ_n, K_n and γ_n values as stated in each Theorem. From the definition of $\mathcal{Z}(\lambda_n;K_n,\gamma_n)$, we have

$$\mathcal{Z}(\lambda_n;K_n,\gamma_n) = \bigcap_{S\in\mathcal{P}_n:\ \lambda_n\le|S|\le\lfloor\frac{n-\gamma_n}{2}\rfloor}\left(\mathcal{E}_n(K_n,\gamma_n;S)\right)^{\mathrm{c}},$$

where $\mathcal{P}_n$ is the collection of all non-empty subsets of $\mathcal{N}_R$. Complementing both sides and using the union bound, we get

$$\begin{aligned}\mathbb{P}\left[(\mathcal{Z}(\lambda_n;K_n,\gamma_n))^{\mathrm{c}}\right] &\le \sum_{S\in\mathcal{P}_n:\ \lambda_n\le|S|\le\lfloor\frac{n-\gamma_n}{2}\rfloor}\mathbb{P}[\mathcal{E}_n(K_n,\gamma_n;S)]\\ &= \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\sum_{S\in\mathcal{P}_{n;r}}\mathbb{P}[\mathcal{E}_n(K_n,\gamma_n;S)], \quad (8)\end{aligned}$$

where $\mathcal{P}_{n;r}$ denotes the collection of all subsets of $\mathcal{N}_R$ with exactly r elements. For each $r=1,\ldots,\lfloor(n-\gamma_n)/2\rfloor$, we can simplify the notation by denoting $\mathcal{E}_{n;r}(K_n,\gamma_n)=\mathcal{E}_n(K_n,\gamma_n;\{1,\ldots,r\})$. From the exchangeability of the node labels and associated random variables, we have

$$\mathbb{P}[\mathcal{E}_n(K_n,\gamma_n;S)] = \mathbb{P}[\mathcal{E}_{n;r}(K_n,\gamma_n)], \quad S\in\mathcal{P}_{n;r}.$$

$|\mathcal{P}_{n;r}| = \binom{n-\gamma_n}{r}$, since there are $\binom{n-\gamma_n}{r}$ subsets of $\mathcal{N}_R$ with r elements. Thus, we have

$$\sum_{S\in\mathcal{P}_{n;r}}\mathbb{P}[\mathcal{E}_n(K_n,\gamma_n;S)] = \binom{n-\gamma_n}{r}\mathbb{P}[\mathcal{E}_{n;r}(K_n,\gamma_n)].$$

Substituting this into (8), we obtain

$$\mathbb{P}\left[(\mathcal{Z}(\lambda_n;K_n,\gamma_n))^{\mathrm{c}}\right] \le \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\binom{n-\gamma_n}{r}\mathbb{P}[\mathcal{E}_{n;r}(K_n,\gamma_n)] \quad (9)$$

Remember that $\mathcal{E}_{n;r}(K_n,\gamma_n)$ is the event that the $n-\gamma_n-r$ nodes in S and r nodes in S^{c} do not pick each other, but they can pick nodes from the set $\mathcal{N}_D$. Thus, we have:

$$\begin{aligned}\mathbb{P}[\mathcal{E}_{n;r}(K_n,\gamma_n)] &= \left(\frac{\binom{\gamma_n+r-1}{K_n}}{\binom{n-1}{K_n}}\right)^r\left(\frac{\binom{n-r-1}{K_n}}{\binom{n-1}{K_n}}\right)^{n-\gamma_n-r} \quad (10)\\ &\le \left(\frac{\gamma_n+r}{n}\right)^{rK_n}\left(\frac{n-r}{n}\right)^{K_n(n-\gamma_n-r)}\end{aligned}$$

Abbreviating $\mathbb{P}\left[\mathcal{Z}(1;K_n,\gamma_n)^{\mathrm{c}}\right]$ as P_Z, we get from (9) that

$$P_Z \le \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\binom{n-\gamma_n}{r}\left(\frac{\gamma_n+r}{n}\right)^{rK_n}\left(\frac{n-r}{n}\right)^{K_n(n-\gamma_n-r)} \quad (11)$$

Using the upper bound on binomials (14) again, we have

$$\begin{aligned}P_Z \le \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}&\left(\frac{n-\gamma_n}{r}\right)^r\left(\frac{n-\gamma_n}{n-\gamma_n-r}\right)^{n-\gamma_n-r}\\ &\cdot\left(\frac{\gamma_n+r}{n}\right)^{rK_n}\left(\frac{n-r}{n}\right)^{K_n(n-\gamma_n-r)} \quad (12)\end{aligned}$$

In order to establish the Theorems, we need to show that (12) goes to zero in the limit of large n for λ_n, γ_n and K_n values as specified in each Theorem.

Since they will be referred to frequently throughout the proofs, we also include here the following standard bounds.

$$1\pm x\le e^{\pm x} \quad (13)$$

$$\binom{n}{m}\le\left(\frac{n}{m}\right)^m\left(\frac{n}{n-m}\right)^{n-m}, \quad \forall m=1,\ldots,n \quad (14)$$

B. A Proof of Theorem 3.1

Recall that in Theorem 3.1, we have $\gamma_n=\alpha n$ with $0<\alpha<1$ and that we need $\lambda_n=1$ for connectivity. Using (13) in (12), we have

$$P_Z \le \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\left(\frac{n-\alpha n}{r}\right)^r e^r\left(\alpha+\frac{r}{n}\right)^{rK_n}e^{-\frac{rK_n(n-\gamma_n-r)}{n}}$$

We will show that the right side of the above expression goes to zero as n goes to infinity. Let

$$A_{n,r,\alpha} := \left(\frac{n-\alpha n}{r}\right)^r e^r \left(\alpha+\frac{r}{n}\right)^{rK_n} e^{-\frac{rK_n(n-\alpha n-r)}{n}}.$$

We write

$$P_Z \le \sum_{r=1}^{\lfloor n/\log n\rfloor} A_{n,r,\alpha} + \sum_{r=\lfloor n/\log n\rfloor}^{\lfloor\frac{n-\alpha n}{2}\rfloor} A_{n,r,\alpha} := S_1+S_2,$$

and show that both S_1 and S_2 go to zero as $n\to\infty$. We start with the first summation S_1.

$$S_1 \le \sum_{r=1}^{\lfloor n/\log n\rfloor}\left((1-\alpha)en\cdot e^{K_n\log(\alpha+\frac{1}{\log n})-K_n(1-\alpha-\frac{1}{\log n})}\right)^r$$

Next, assume as in the statement of Theorem 3.1 that

$$K_n = \frac{c_n\log n}{1-\alpha-\log\alpha},\quad n=1,2,\ldots \tag{15}$$

for some sequence $c:\mathbb{N}_0\to\mathbb{R}_+$ such that $\lim_{n\to\infty}c_n=c$ with $c>1$. Also define

$$\begin{aligned}
a_n &:= (1-\alpha)en\cdot e^{K_n\log(\alpha+\frac{1}{\log n})-K_n(1-\alpha-\frac{1}{\log n})}\\
&= (1-\alpha)en\cdot e^{-\frac{c_n\log n}{1-\alpha-\log\alpha}\left(1-\alpha-\frac{1}{\log n}-\log(\alpha+\frac{1}{\log n})\right)}\\
&= (1-\alpha)en^{1-c_n}\cdot e^{\frac{c_n}{1-\alpha-\log\alpha}\left(1-\log n\log(1+\frac{1}{\alpha\log n})\right)}\\
&= O(1)n^{1-c_n}
\end{aligned}$$

where we substituted K_n via (15) and used the fact that $\log n\cdot\log(1+\frac{1}{\alpha\log n})=\frac{1}{\alpha}+o(1)$. Taking the limit as $n\to\infty$ and recalling that $\lim_{n\to\infty}c_n=c>1$, we see that $\lim_{n\to\infty}a_n=0$. Hence, for large n, we have

$$S_1 \le \sum_{r=1}^{\lfloor n/\log n\rfloor}(a_n)^r \le \sum_{r=1}^{\infty}(a_n)^r = \frac{a_n}{1-a_n} \tag{16}$$

where the geometric sum converges by virtue of $\lim_{n\to\infty}a_n=0$. Using this, it is clear that $\lim_{n\to\infty}S_1=0$.

Now, consider the second summation S_2.

$$\begin{aligned}
S_2 &\le \sum_{r=\lfloor n/\log n\rfloor}^{\lfloor(n-\alpha n)/2\rfloor}\left(\frac{(n-\alpha n)e}{n/\log n}\right)^r\left(\frac{\alpha n+\frac{n-\alpha n}{2}}{n}\right)^{rK_n}\\
&\qquad\cdot e^{-\frac{rK_n}{n}(n-\alpha n-\frac{n-\alpha n}{2})}\\
&\le \sum_{r=\lfloor n/\log n\rfloor}^{\lfloor(n-\alpha n)/2\rfloor}\left((1-\alpha)e\log n\cdot e^{K_n\log(\frac{1+\alpha}{2})-K_n\frac{1-\alpha}{2}}\right)^r
\end{aligned}$$

Next, we define

$$\begin{aligned}
b_n &:= (1-\alpha)e\log n\cdot e^{-K_n\left(\frac{1-\alpha}{2}-\log(\frac{1+\alpha}{2})\right)} &(17)\\
&= (1-\alpha)e\log n\cdot e^{-\frac{c_n\log n}{1-\alpha-\log\alpha}\left(\frac{1-\alpha}{2}-\log(\frac{1+\alpha}{2})\right)} &(18)
\end{aligned}$$

where we substituted for K_n via (15). Taking the limit as $n\to\infty$ we see that $\lim_{n\to\infty}b_n=0$ upon noting that $\frac{1-\alpha}{2}-\log(\frac{1+\alpha}{2})>0$ and $\lim_{n\to\infty}c_n=c>1$. With arguments similar to those used in the case of S_1, we can show that when n is large, $S_2\le b_n/(1-b_n)$, leading to S_2 converging to zero as n gets large. With $P_Z\le S_1+S_2$, and both S_1 and S_2 converging to zero when n is large, we establish the fact that P_Z converges to zero as n goes to infinity. This result also yields the desired conclusion $\lim_{n\to\infty}P(n,K_n,\gamma_n)=1$ in Theorem 3.1 since $P_Z=1-P(n,K_n,\gamma_n)$.

C. A Proof of Theorem 3.2

We will first start with part (a) of Theorem 3.2.

Part a) Recall that in part (a), $\gamma_n=o(\sqrt{n})$ and we need $\lambda_n=1$ for connectivity. Using this and (13) in (12), we get

$$\begin{aligned}
P_Z &\le \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\left(\frac{n-\gamma_n}{r}\right)^r\left(\frac{n-\gamma_n}{n-\gamma_n-r}\right)^{n-\gamma_n-r}\\
&\qquad\cdot\left(\frac{\gamma_n+r}{n}\right)^{rK_n}\left(\frac{n-r}{n}\right)^{K_n(n-\gamma_n-r)}\\
&\le \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\left(\frac{n-\gamma_n}{r}\right)^r\left(1+\frac{r\gamma_n}{n(n-\gamma_n-r)}\right)^{n-\gamma_n-r}\\
&\qquad\cdot\left(\frac{\gamma_n+r}{n}\right)^{rK_n}\left(\frac{n-r}{n}\right)^{(K_n-1)(n-\gamma_n-r)}\\
&\le \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}\left(1+\frac{\gamma_n}{r}\right)^r\left(\frac{\gamma_n+r}{n}\right)^{r(K_n-1)}\\
&\qquad\cdot e^{-\frac{r(K_n-1)(n-\gamma_n-r)}{n}}
\end{aligned}$$

We will show that the right side of the above expression goes to zero as n goes to infinity. Let

$$A_{n,r,\gamma_n} := \left(1+\frac{\gamma_n}{r}\right)^r\left(\frac{\gamma_n+r}{n}\right)^{r(K_n-1)} e^{-\frac{r(K_n-1)(n-\gamma_n-r)}{n}}$$

We write

$$P_Z \le \sum_{r=1}^{\lfloor\sqrt{n}\rfloor}A_{n,r,\gamma_n}+\sum_{r=\lceil\sqrt{n}\rceil}^{\lfloor\frac{n-\gamma_n}{2}\rfloor}A_{n,r,\gamma_n} := S_1+S_2,$$

and show that both S_1 and S_2 go to zero as $n\to\infty$. We start with the first summation S_1.

$$\begin{aligned}
S_1 &\le \sum_{r=1}^{\lfloor\sqrt{n}\rfloor}\left(1+\frac{\gamma_n}{r}\right)^r\left(\frac{\gamma_n+r}{n}\right)^{r(K_n-1)} e^{-\frac{r(K_n-1)(n-\gamma_n-r)}{n}}\\
&\le \sum_{r=1}^{\lfloor\sqrt{n}\rfloor}\left(e^{\log(1+\gamma_n)+(K_n-1)\left[\log\left(\frac{\gamma_n+\sqrt{n}}{n}\right)-\frac{n-\gamma_n-\sqrt{n}}{n}\right]}\right)^r
\end{aligned}$$

Next, assume as in the statement of Theorem 3.2(a) that $K_n\ge 2,\ \forall n$. Also define

$$\begin{aligned}
a_n &:= e^{\log(1+\gamma_n)+(K_n-1)\left[\log\left(\frac{\gamma_n+\sqrt{n}}{n}\right)-\frac{n-\gamma_n-\sqrt{n}}{n}\right]}\\
&\le e^{\log(1+\gamma_n)+\log\left(1+\frac{\gamma_n}{\sqrt{n}}\right)-\log(\sqrt{n})}e^{-\frac{n-\gamma_n-\sqrt{n}}{n}}\\
&= O(1)e^{\log(1+\gamma_n)-\log(\sqrt{n})}
\end{aligned}$$

Taking the limit as $n\to\infty$ and recalling that $\gamma_n=o(\sqrt{n})$, we see that $\lim_{n\to\infty}a_n=0$. Hence, for large n, we have

$$S_1 \le \sum_{r=1}^{\lfloor\sqrt{n}\rfloor}(a_n)^r \le \sum_{r=1}^{\infty}(a_n)^r = \frac{a_n}{1-a_n} \tag{19}$$

where the geometric sum converges by virtue of $\lim_{n\to\infty} a_n = 0$. Using this once again, it is clear from the last expression that $\lim_{n\to\infty} S_1 = 0$.

Now, consider the second summation S_2.

$$S_2 \le \sum_{r=\lceil\sqrt{n}\rceil}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \left(e^{\frac{\sqrt{n}}{n}+(K_n-1)\left[\log\left(\frac{n+\gamma_n}{2n}\right)-\frac{n-\gamma_n}{2n}\right]}\right)^r$$

Again assume as in the statement of Theorem 3.2(a) that $K_n \ge 2$. Next, we define=

$$\begin{aligned} b_n &:= e^{\frac{\sqrt{n}}{n}+(K_n-1)\left[\log\left(\frac{n+\gamma_n}{2n}\right)-\frac{n-\gamma_n}{2n}\right]} \\ &\le e^{\frac{\sqrt{n}}{n}+\log\left(\frac{1}{2}\right)+\log\left(\frac{n}{n+\gamma_n}\right)-\frac{1}{2}+\frac{\gamma_n}{2n}} \\ &= O(1)e^{-\log\left(\sqrt{n}\right)} \end{aligned}$$

Taking the limit as $n \to \infty$ and recalling that $\gamma_n = o(\sqrt{n})$, we see that $\lim_{n\to\infty} b_n = 0$. Hence, for large n, we have

$$S_2 \le \sum_{r=\lceil\sqrt{n}\rceil}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} (b_n)^r \le \sum_{r=\lceil\sqrt{n}\rceil}^{\infty} (b_n)^r = \frac{(b_n)^{\sqrt{n}}}{1-b_n} \tag{20}$$

where the geometric sum converges by virtue of $\lim_{n\to\infty} b_n = 0$. Using this once again, it is clear from the last expression that $\lim_{n\to\infty} S_2 = 0$. With $P_Z \le S_1 + S_2$, and both S_1 and S_2 converging to zero when n is large, we establish the fact that P_Z converges to zero as n goes to infinity. This result also yields the desired conclusion $\lim_{n\to\infty} P(n, K_n, \gamma_n) = 1$ in Theorem 3.2(a) since $P_Z = 1 - P(n, K_n, \gamma_n)$.

Part b) We now continue with the proof of Theorem 3.2(b). Recall that we had $\gamma_n = \Omega(\sqrt{n})$ and $\gamma_n = o(n)$. Using this and (13) in (12), we get

$$\begin{aligned} P_Z \le& \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \left(\frac{n-\gamma_n}{r}\right)^r \left(\frac{n-\gamma_n}{n-\gamma_n-r}\right)^{n-\gamma_n-r} \\ &\cdot \left(\frac{\gamma_n+r}{n}\right)^{rK_n} e^{-\frac{rK_n(n-\gamma_n-r)}{n}} \\ \le& \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} e^{(1-\gamma_n/n)r} \left(\frac{\gamma_n+r}{r}\right)^r \left(\frac{\gamma_n+r}{n}\right)^{r(K_n-1)} \\ &\cdot e^{-\frac{rK_n(n-\gamma_n-r)}{n}} \\ \le& \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \exp\left(r\left[(K_n-1)\left(\log\left(\frac{\gamma_n+r}{n}\right)+\frac{r}{n}\right.\right.\right. \\ &\left.\left.\left.+\frac{\gamma_n}{n}-1\right)+\log\left(1+\gamma_n\right)+\frac{n-\gamma_n}{2n}\right]\right) \end{aligned} \tag{21}$$

Next, assume as in the statement of Theorem 3.2(b) that

$$K_n = \frac{\log(\gamma_n+1)}{\log 2 + 1/2} + w(1), \quad n = 1, 2, \ldots \tag{22}$$

Since $K_n - 1 > 0, \quad \forall n = 1, 2, \ldots$, and noting that $r \le \frac{n-\gamma_n}{2}$ in (21), we have

$$\begin{aligned} &(K_n-1)\left(\log\left(\frac{\gamma_n+r}{n}\right)+\frac{r}{n}+\frac{\gamma_n}{n}-1\right) \le \\ &(K_n-1)\left(\log\left(\frac{\gamma_n+\frac{n-\gamma_n}{2}}{n}\right)+\frac{\frac{n-\gamma_n}{2}}{n}+\frac{\gamma_n}{n}-1\right) \end{aligned} \tag{23}$$

Using this, we get

$$\begin{aligned} P_Z \le \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \exp\left(r\left[(K_n-1)\left(\log\left(\frac{n+\gamma_n}{2n}\right)\right.\right.\right. \\ \left.\left.\left.-\frac{n-\gamma_n}{2n}\right)+\log\left(1+\gamma_n\right)+\frac{n-\gamma_n}{2n}\right]\right) \end{aligned} \tag{24}$$

Next, define

$$a_n := e^{(K_n-1)\left(\log\left(\frac{n+\gamma_n}{2n}\right)-\frac{n-\gamma_n}{2n}\right)+\log(1+\gamma_n)+\frac{n-\gamma_n}{2n}} \tag{25}$$

Recall that $\gamma_n = o(n)$, so we have $\lim_{n\to\infty} \gamma_n/n = 0$. Using this, and substituting K_n via (22), we get

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left[e^{\left(\frac{\log(\gamma_n+1)}{\log 2+1/2}+w(1)\right)\left(-\log 2-\frac{1}{2}\right)+\log(1+\gamma_n)+\frac{1}{2}}\right] \\ &= \lim_{n\to\infty} \left[e^{-w(1)(\log 2+1/2)-\log(1+\gamma_n)+\log(1+\gamma_n)+\frac{1}{2}}\right] \\ &= \lim_{n\to\infty} \left[o(1)e^{-w(1)(\log 2+1/2)}\right] = 0 \end{aligned} \tag{26}$$

Hence, for large n, we have

$$P_Z \le \sum_{r=1}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} (a_n)^r \le \sum_{r=1}^{\infty} (a_n)^r = \frac{a_n}{1-a_n} \tag{27}$$

where the geometric sum converges by virtue of $\lim_{n\to\infty} a_n = 0$. Using this, it is clear from the last expression that $\lim_{n\to\infty} P_Z = 0$. This result also yields the desired conclusion $\lim_{n\to\infty} P(n, K_n, \gamma_n) = 1$ in Theorem 3.2(b) since $P_Z = 1 - P(n, K_n, \gamma_n)$. This result, combined with the proof of part a, concludes the proof of Theorem 3.2.

D. A Proof of Theorem 3.3

Recall that in Theorem 3.3, we have $\gamma_n = o(n)$ and $\lambda_n = \Omega(\sqrt{n})$. Using (13) in (12), we have

$$\begin{aligned} P_Z \le& \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \left(\frac{n-\gamma_n}{r}\right)^r \left(\frac{n-\gamma_n}{n-\gamma_n-r}\right)^{n-\gamma_n-r} \\ &\cdot \left(\frac{\gamma_n+r}{n}\right)^{rK_n} \left(\frac{n-r}{n}\right)^{K_n(n-\gamma_n-r)} \\ \le& \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \left(\frac{n-\gamma_n}{r}\right)^r \left(1+\frac{r\gamma_n}{n(n-\gamma_n-r)}\right)^{n-\gamma_n-r} \\ &\cdot \left(\frac{\gamma_n+r}{n}\right)^{rK_n} \left(\frac{n-r}{n}\right)^{(K_n-1)(n-\gamma_n-r)} \\ \le& \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \left(\frac{n-\gamma_n}{r}\right)^r e^{\frac{r\gamma_n}{n}} \left(\frac{\gamma_n+r}{n}\right)^r \\ &\cdot \left(\frac{\gamma_n+r}{n}\right)^{r(K_n-1)} e^{-\frac{r(K_n-1)(n-\gamma_n-r)}{n}} \\ \le& \sum_{r=\lambda_n}^{\lfloor\frac{n-\gamma_n}{2}\rfloor} \left(1+\frac{\gamma_n}{r}\right)^r \left(\frac{\gamma_n+r}{n}\right)^{r(K_n-1)} \\ &\cdot e^{-\frac{r(K_n-1)(n-\gamma_n-r)}{n}} \end{aligned}$$

Next, assume as in the statement of Theorem 3.3 that

$$K_n > 1 + \frac{\log(1+\gamma_n/\lambda_n)}{\log 2 + 1/2} \quad (28)$$

Since $K_n > 1$, we have

$$P_Z \leq \sum_{r=n}^{\lfloor \frac{n}{n} \rfloor} \left(1 + \frac{\gamma_n}{\lambda_n}\right)^r \left(\frac{n+\gamma_n}{2n}\right)^{r(K_n-1)} \cdot e^{\frac{r(K_n-1)(n-n)}{2n}}$$

We will show that the right side of the above expression goes to zero as n goes to infinity. Let

$$a_n := e^{\log\left(1+\frac{n}{n}\right)+(K_n-1)\left[\log\left(\frac{n+n}{2n}\right)-\frac{n-n}{2n}\right]}$$

Recall that $\gamma_n = o(n)$, so we have $\lim_{n\to\infty} \gamma_n/n = 0$. Using this, and substituting for K_n via (28), we get

$$a_n < e^{\log\left(1+\frac{n}{n}\right)+\left(\frac{\log(1+n/n)}{\log(1/2)-1/2}\right)\left[\log\left(\frac{n+n}{2n}\right)-\frac{n-n}{2n}\right]} \quad (29)$$

Taking the limit $n \to \infty$, we have

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} e^{\log\left(1+\frac{n}{n}\right)-\log(1+n/n)} = e^0 = 1 \quad (30)$$

Hence, for large n, we have

$$P_Z \leq \sum_{r=n}^{\lfloor \frac{n}{n} \rfloor} (a_n)^r \leq \sum_{r=n}^{\infty} (a_n)^r = \frac{(a_n)^{n}}{1-a_n} \quad (31)$$

where the geometric sum converges by virtue of $\lim_{n\to\infty} a_n < 1$ and $\lim_{n\to\infty} \lambda_n = w(1)$. Using this, it is clear from the last expression that $\lim_{n\to\infty} P_Z = 0$. This result also yields the desired conclusion $\lim_{n\to\infty} P_G(n, K_n, \gamma_n, \lambda_n) = 1$ in Theorem 3.3 since $P_Z = 1 - P_G(n, K_n, \gamma_n, \lambda_n)$. This concludes the proof of Theorem 3.3.

E. A Proof of Theorem 3.4

Recall that in Theorem 3.4, we have $\gamma_n = \alpha n$ with α in $(0,1)$, and $\lambda_n < \frac{(1-\alpha)n}{2}$. Using $\gamma_n = \alpha n$ in (12), we get

$$\begin{aligned} P_Z \leq & \sum_{r=n}^{\lfloor \frac{n}{n} \rfloor} \left(\frac{n-\alpha n}{r}\right)^r \left(\frac{n-\alpha n}{n-\alpha n-r}\right)^{n-n-r} \\ & \cdot \left(\frac{\alpha n+r}{n}\right)^{rK_n} \left(\frac{n-r}{n}\right)^{K_n(n-n-r)} \\ \leq & \sum_{r=n}^{\lfloor \frac{n}{n} \rfloor} (1-\alpha)^r e^{r} \left(1+\frac{\alpha n}{r}\right)^r \\ & \cdot \left(\frac{\alpha n+r}{n}\right)^{r(K_n-1)} e^{r(K_n-1)\frac{(n-n-r)}{n}} \end{aligned} \quad (32)$$

Next, assume as in the statement of Theorem 3.4 that

$$K_n > 1 + \frac{\log(1+\frac{n}{n}) + \alpha + \log(1-\alpha)}{\frac{1}{2} + \log 2 - \log(1+\alpha)}, \quad n = 1, 2, \ldots$$

Since $K_n > 1$, we have

$$P_Z \leq \sum_{r=n}^{\lfloor \frac{n}{n} \rfloor} (1-\alpha)^r e^{r} \left(1+\frac{\alpha n}{\lambda_n}\right)^r \left(\frac{1+\alpha}{2}\right)^{r(K_n-1)} \cdot e^{r(K_n-1)\left(\frac{1}{2}\right)}$$

Also define

$$\begin{aligned} a_n &:= e^{+\log(1-)+\log(1+\frac{n}{n})+(K_n-1)\left[\log\left(\frac{1+}{2}\right)-\left(\frac{1}{2}\right)\right]} \\ &< e^{+\log(1-)+\log(1+\frac{n}{n})-\left(+\log(1-)+\log(1+\frac{n}{n})\right)} = 1 \end{aligned}$$

where we substituted K_n via (33). Taking the limit as $n \to \infty$, we see that $\lim_{n\to\infty} a_n < 1$. Hence, for large n, we have

$$P_Z \leq \sum_{r=n}^{\lfloor \frac{n}{n} \rfloor} (a_n)^r \leq \sum_{r=n}^{\infty} (a_n)^r = \frac{(a_n)^{n}}{1-a_n} \quad (33)$$

where the geometric sum converges by virtue of $\lim_{n\to\infty} a_n < 1$ and $\lim_{n\to\infty} \lambda_n = w(1)$. Using this, it is clear from the last expression that $\lim_{n\to\infty} P_Z = 0$. This result also yields the desired conclusion $\lim_{n\to\infty} P_G(n, K_n, \gamma_n, \lambda_n) = 1$ in Theorem 3.4 since $P_Z = 1 - P_G(n, K_n, \gamma_n, \lambda_n)$. This concludes the proof of Theorem 3.4.

F. Preliminaries Needed in the Proof of Theorem 3.5

We start with a few definitions and properties that will be useful throughout the rest of the proof. First, let $B(a,b)$ denote the beta function, $B_x(a,b)$ denote the incomplete beta function, and $I_x(a,b)$ denote the regularized incomplete beta function, where a and b are non-negative integers. These functions are defined as follows [30]:

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{(a-1)!(b-1)!}{(a+b-1)!} \\ B_x(a,b) &= \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad 0 \leq x \leq 1 \\ I_x(a,b) &= \frac{B_x(a,b)}{B(a,b)}, \quad 0 \leq x \leq 1 \end{aligned} \quad (34)$$

Using these definitions, it can easily be shown that

$$I_{1/2}(r,r) = 1/2, \quad r > 0 \quad (35)$$

Proof: $$\begin{aligned} B(r,r) &= \int_0^1 t^{r-1}(1-t)^{r-1} dt \\ &= 2\int_0^{\frac{1}{2}} t^{r-1}(1-t)^{r-1} dt = 2B_{1/2}(r,r) \end{aligned}$$

where we divided the integral into two parts since the function $(t-t^2)^{r-1}$ is symmetric around 1/2. Using the fact that $B_{1/2}(r,r) = I_{1/2}(r,r)B(r,r)$, we can conclude that $I_{1/2}(r,r) = 1/2$.

The cumulative distribution function $F(a;n,p)$ of a Binomial random variable $X \sim B(n,p)$ can be expressed using the regularized incomplete beta function as:

$$\begin{aligned} F(a;n,p) &= \mathbb{P}[X \leq a] = I_{1-p}(n-a, a+1) \\ &= (n-a)\binom{n}{a}\int_0^{1-p} t^{n-a-1}(1-t)^a dt \end{aligned} \quad (36)$$

Lemma 4.2: [30, Eq. 8.17.4]: For $a, b > 0,\ 0 \le x \le 1$,

$$I_x(a,b) = 1 - I_{1-x}(b,a) \tag{37}$$

Lemma 4.3: [30, Eq. 8.17.20]: For $a, b > 0,\ 0 \le x \le 1$,

$$I_x(a+1,b) = I_x(a,b) - \frac{x^a(1-x)^b}{aB(a,b)} \tag{38}$$

Lemma 4.4: The equation $I_\alpha(r,r) = c\alpha$ has only one solution when $r > 0$, $r \in \mathbb{Z}^+$, $0 < \alpha \le 1/2$ and $0 < c \le 1$.

Proof: First, $\alpha = 0$ is a solution to this equation since when $\alpha = 0$, both $I_\alpha(r,r)$ and $c\alpha$ are zero. Also, when $c = 1$, $\alpha = 1/2$ is a solution of the equation since $I_{1/2}(r,r) = 1/2$. The derivative of both terms with respect to α is:

$$\frac{\partial(I_\alpha(r,r))}{\partial\alpha} = \frac{\alpha^{r-1}(1-\alpha)^{r-1}}{B(r,r)}, \quad \frac{\partial(c\alpha)}{\partial\alpha} = c \tag{39}$$

It can be seen that the derivative of $c\alpha$ is a constant, and $\frac{(\alpha(1-\alpha))^{r-1}}{B(r,r)} = 0$ when $\alpha = 0$ and $\frac{(\alpha(1-\alpha))^{r-1}}{B(r,r)}$ is monotone increasing in the range $0 < \alpha \le 1/2$. For the case where $c = 1$; $\alpha = 0$ and $\alpha = 1/2$ is a solution to the equation, hence for some $0 < \alpha^* < 1/2$ that satisfies $\frac{(\alpha^*(1-\alpha^*))^{r-1}}{B(r,r)} = 1$, it must hold that $\frac{\partial(I_\alpha(r,r))}{\partial\alpha} < \frac{\partial(\alpha)}{\partial\alpha} = 1$ when $0 < \alpha < \alpha^*$, and $\frac{\partial(I_\alpha(r,r))}{\partial\alpha} > \frac{\partial(\alpha)}{\partial\alpha} = 1$ when $\alpha^* < \alpha \le 1/2$. This is because if such α^* such that $0 < \alpha^* < 1/2$ does not exist, $\alpha = 1/2$ can't be a solution to the equation $I_\alpha(r,r) = c\alpha$. Now, considering the case for arbitrary $0 < c \le 1$, since $\frac{\alpha^r(1-\alpha)^r}{B(r,r)} = 0$ is monotone increasing, there can only be one $0 < \alpha^* < 1/2$ such that $\frac{(\alpha^*)^r(1-\alpha^*)^r}{B(r,r)} = c$. This means that $c\alpha$ is increasing faster than $I_\alpha(r,r)$ in the region $0 < \alpha < \alpha^*$, hence there can't be a solution to $I_\alpha(r,r) = c\alpha$ in this region. Further, $I_\alpha(r,r)$ is increasing faster than $c\alpha$ in the region $\alpha^* < \alpha \le 1/2$, hence there can be at most one solution to the equation $I_\alpha(r,r) = c\alpha$ in the region $0 < \alpha \le 1/2$. Now, consider the fact that $I_{\alpha^*}(r,r) < c\alpha^*$, and $I_{1/2}(r,r) = 1/2 \ge c/2$ when $\alpha = 1/2$. Combining this with the fact that both functions are continuous, there must be at least one solution to the equation $I_\alpha(r,r) = c\alpha$ for $0 < c \le 1$ in the range $\alpha^* < \alpha \le 1/2$. Combining this with previous statement (that there can be at most one solution in this range), it can be concluded that there is only one solution to the equation $I_\alpha(r,r) = c\alpha$ in the range $0 < \alpha \le 1/2$ where $0 < c < 1$.

G. Proof of Theorem 3.5

To prove Theorem 3.5, we need to show that for any $r \in \mathbb{Z}^+$, the random K-out graph $\mathbb{H}(n; K_n)$ is r-robust *whp* if $K_n \ge 2r$. To do this, similar to the proof given in [28] for Erdős-Rényi graphs, we will first find an upper bound on the probability of a subset of given size being not r-reachable, and then use this result to show that the probability of not being r-robust goes to zero when $n \to \infty$ and $K_n \ge 2r$. Different from the prior work which relied on the commonly used upper bounds for the binomial coefficients $\binom{n}{k} \le \left(\frac{en}{k}\right)^k$ and the union bound [2], [28] to bound the probability of a subset of given size being not r-reachable, our proof uses the Beta function $B(a,b)$ and its properties described in the previous Section to achieve tighter bounds. This in turn enables us to establish a tighter threshold for the r-robustness of random K-out graphs than what was previously possible; e.g., see [2].

First, let $\mathcal{E}_n(K_n, r; S)$ denote the event that $S \subset V$ is an r-reachable set as per Definition 2.7. The event $\mathcal{E}_n(K_n, r; S)$ occurs if there exists at least one node in S that is adjacent to at least r nodes in S^c, the subset comprised of nodes outside the subset S. Thus, we have

$$\mathcal{E}_n(K_n, r; S) = \bigcup_{i \in \mathcal{N}_S} \left\{ \left(\sum_{j \in \mathcal{N}_R} \mathbb{1}\{v_i \sim v_j\} \right) \ge r \right\},$$

with $\mathcal{N}_S$, $\mathcal{N}_{S^c}$ denoting the set of labels of the vertices in S and S^c, respectively, and $\mathbb{1}\{\}$ denoting the indicator function. We are also interested in the complement of this event, denoted as $(\mathcal{E}_n(K_n, r; S))^c$, which occurs if all nodes in S are adjacent to less than r nodes in S^c. This can be written as

$$(\mathcal{E}_n^c(K_n, r; S)) = \bigcap_{i \in \mathcal{N}_S} \left\{ \left(\sum_{j \in \mathcal{N}_{S^c}} \mathbb{1}\{v_i \sim v_j\} \right) < r \right\}.$$

Note that at least one subset in every disjoint subset pairs that partition V needs to be r-reachable per the definition of r-robustness, hence one of the events $\mathcal{E}_n(K_n, r; S)$ or $\mathcal{E}_n(K_n, r; \mathcal{S}^c)$ need to hold *with high probability* for every subset S of V. Now, let $\mathcal{Z}(K_n, r)$ denote the event that none of the subsets in all subset pairs S and S^c such that $S \subset V$ are r-reachable. Thus, we have

$$\mathcal{Z}(K_n, r) = \bigcup_{S \in \mathcal{P}_n :\ |S| \le \lfloor \frac{n}{2} \rfloor} \left[(\mathcal{E}_n(K_n, r; S))^c \wedge (\mathcal{E}_n(K_n, r; S^c))^c \right],$$

where $\mathcal{P}_n$ is the collection of all non-empty subsets of V and since for each S we check the r-reachability of both S and S^c, the condition $|S| \le \lfloor \frac{n}{2} \rfloor$ is used to prevent counting each subset twice. Using union bound, we get

$$\begin{aligned} P_Z &\le \sum_{|S| \le \lfloor \frac{n}{2} \rfloor} \mathbb{P}[(\mathcal{E}_n(K_n, r; S))^c \wedge (\mathcal{E}_n(K_n, r; S^c))^c] \\ &= \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{S_m \in \mathcal{P}_{n;m}} \mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^c \wedge (\mathcal{E}_n(K_n, r; S_m^c))^c], \end{aligned} \tag{40}$$

where $\mathcal{P}_{n;m}$ denotes the collection of all subsets of V with exactly m elements, and let $S_m \in \mathcal{P}_{n;m}$ is a subset of the vertex set V with size m, i.e. $S_m \subset V$ and $|S_m| = m$. Further, $\mathbb{P}[\mathcal{Z}(K_n, r)]$ is abbreviated as $P_Z := \mathbb{P}[\mathcal{Z}(K_n, r)]$, and $S_m^c = V \setminus S_m$. From the exchangeability of the node labels and associated random variables, we have

$$\begin{aligned} &\sum_{S_m \in \mathcal{P}_{n;m}} \mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^c \wedge (\mathcal{E}_n(K_n, r; S_m^c))^c] \\ &= \binom{n}{m} \mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^c \wedge (\mathcal{E}_n(K_n, r; S_m^c))^c]. \end{aligned} \tag{41}$$

since $|\mathcal{P}_{n;m}| = \binom{n}{m}$, as there are $\binom{n}{m}$ subsets of V with m elements. Substituting this into (40), we obtain

$$P_Z \le \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{m} \mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^c \wedge (\mathcal{E}_n(K_n, r; S_m^c))^c]$$

Before evaluating this expression, we will start with evaluating $\mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^{\mathsf{c}}]$, the probability that the set S_m is not r-robust. Let P_{S_m} denote the probability that each node in S_m chooses, and forms an edge with less than r nodes in S_m^{c} during the construction of the random K-out graph. Since this does not take the selections of the nodes in S_m^{c} into account, $\mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^{\mathsf{c}}] \leq P_{S_m}$. Further, let $v \in S_m$ be a node in the set S_m.

Lemma 4.5: The probability that the node $v \in S_m$ chooses less than r nodes in the set S_m^{c}, denoted as P_v, can be upper bounded by the cumulative distribution function $F(r-1; K_n, p)$ of a binomial random variable with K_n trials and success probability $p = \frac{n-m-r+1}{n-r}$.

Proof: For node v, after making one selection, the number of nodes available to choose from decreases so the probability of choosing a node in S_m^{c} changes at each selection. For example, the probability of choosing a node in S_m^{c} in the first selection is $\frac{n-m}{n-1}$ and the probability of choosing a node in S_m^{c} in the second selection is $\frac{n-m-1}{n-2}$ if a node in S_m^{c} was selected in the first selection and it is $\frac{n-m}{n-2}$ otherwise. Based on this, the probability of selecting a node in S_m^{c} at the i^{th} selection out of K_n selections can be expressed as $\frac{n-m-j}{n-i}$, $1 \leq i \leq K_n$, $0 \leq j < i$ where j denotes the number of nodes already chosen from the set S_m^{c} before the i^{th} selection. Since we are considering the case of choosing less than r nodes in S_m^{c}, we have that $j < r$, and with this constraint the lowest possible value of $\frac{n-m-j}{n-i}$ occurs when $j = r-1$ and $i = r$, and hence it is $\frac{n-m-r+1}{n-r}$. This gives a lower bound on the probability of selecting a node in S_m^{c} in one of the K_n selections and at the same time is an upper bound on the probability of not selecting a node in S_m^{c} in one of the K_n selections, and hence it is an upper bound for choosing less than r nodes.

Next, using this upper bound, we plug in $n = K_n$ and $p = \frac{n-m-r+1}{n-r}$ to (36), then we have

$$P_v \leq I_{\frac{m-1}{n-r}}(K_n - r + 1, r) \tag{42}$$

$$= (K_n - r + 1)\binom{K_n}{r-1}\int_0^{\frac{m-1}{n-r}} t^{K_n - r}(1-t)^{r-1}dt$$

Since there are m nodes in the set S_m and the choices of each node are independent, $P_{S_m} = (P_v)^m$. Hence,

$$P_{S_m} \leq \left(I_{\frac{m-1}{n-r}}(K_n - r + 1, r)\right)^m \tag{43}$$

Similarly,

$$\mathbb{P}[(\mathcal{E}_n(K_n, r; S_m^{\mathsf{c}}))^{\mathsf{c}}] \leq P_{S_m^{\mathsf{c}}}$$

$$\leq \left(I_{\frac{n-m-r+1}{n-r}}(K_n - r + 1, r)\right)^{n-m} \tag{44}$$

Since the event in which each node in S_m forms less than r edges with a node in S_m^{c} and the event each node in S_m^{c} forms less than r edges with a node in S_m are independent events (since the selections of each node are independent), the probability of their intersection is their multiplication $P_{S_m}P_{S_m^{\mathsf{c}}}$. Hence,

$$\mathbb{P}[(\mathcal{E}_n(K_n, r; S_m))^{\mathsf{c}} \wedge (\mathcal{E}_n(K_n, r; S_m^{\mathsf{c}}))^{\mathsf{c}}] \leq P_{S_m}P_{S_m^{\mathsf{c}}} \tag{45}$$

Let $P_m := \binom{n}{m} P_{S_m}P_{S_m^{\mathsf{c}}}$. Then, we have

$$P_Z \leq \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{m} P_{S_m}P_{S_m^{\mathsf{c}}} = \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} P_m \tag{46}$$

To find P_m, we first need to find the probability of a node in S_m being adjacent to less than r nodes (less than or equal to $r-1$ nodes) in S_m^{c}, denoted as P_{S_m} and the probability of a node in $\mathcal{R}_m$ being adjacent to less than r nodes in S_m, denoted as $P_{S_m^{\mathsf{c}}}$. We will divide the summation into three parts as follows:

$$P_Z = \sum_{m=1}^{\lfloor n/2 \rfloor} P_m = \sum_{m=1}^{\lfloor \log(n) \rfloor} P_m + \sum_{m=\lceil \log(n) \rceil}^{\lfloor \alpha^*(n-r) \rfloor} P_m$$

$$+ \sum_{m=\lceil \alpha^*(n-r) \rceil}^{\lfloor n/2 \rfloor} P_m = P_1 + P_2 + P_3 \tag{47}$$

where α^* is the solution to the equation $I_{\alpha^*}(r, r) = \frac{n-r}{ne} \cdot \alpha^*$ in the range $0 < \alpha^* < \frac{1}{2}$. (The purpose of defining α^* this way will be given later in the proof.)

Start with the first summation P_1 and use (14) along with $\binom{n}{m} \leq \left(\frac{en}{m}\right)^m$, then we have:

$$P_m \leq \binom{n}{m}(P_{S_m})^m (P_{S_m^{\mathsf{c}}})^{n-m} \leq \binom{n}{m}(P_{S_m})^m$$

$$\leq \left(\frac{en}{mB(K_n - r + 1, r)}\int_0^{\frac{m-1}{n-r}} t^{K_n - r}(1-t)^{r-1}dt\right)^m$$

$$\leq \left(\frac{en\left(\frac{m-1}{n-r}\right)^{K_n - r + 1}}{mB(K_n - r + 1, r)(K_n - r + 1)}\right)^m$$

$$\leq \left(\frac{e(1 + r/m)}{B(K_n - r + 1, r)(K_n - r + 1)}\left(\frac{m-1}{n-r}\right)^{K_n - r}\right)^m$$

$$\leq \left(\frac{e(1+r)\left(\frac{\log(n)-1}{n-r}\right)^{K_n - r}}{B(K_n - r + 1, r)(K_n - r + 1)}\right)^m := (a_n)^m$$

For $K_n > r$, since $B(K_n - r + 1, r)$ and r are finite values, we have $\lim_{n \to \infty} a_n = 0$ by virtue of $\lim_{n \to \infty}\left(\frac{\log(n)-1}{n-r}\right)^{K_n - r} = 0$. Using this, we can express the summation as:

$$P_1 = \sum_{m=1}^{\lfloor \log(n) \rfloor}(a_n)^m \leq a_n \cdot \frac{1 - (a_n)^{\log(n)}}{1 - a_n} \tag{48}$$

where the geometric sum converges by virtue of $\lim_{n \to \infty} a_n = 0$, leading to P_1 converging to zero for large n.

Now, similarly consider the second summation P_2. Using $\binom{n}{m} \leq \left(\frac{en}{m}\right)^m$, we have

$$P_m \leq \binom{n}{m}(P_{S_m})^m (P_{S_m^{\mathsf{c}}})^{n-m} \leq \binom{n}{m}(P_{S_m})^m$$

$$\leq \left(\frac{en}{m} I_{\frac{m-1}{n-r}}(K_n - r + 1, r)\right)^m = (a_m)^m \tag{49}$$

where $a_m := \frac{en}{m} \cdot I_{\frac{m \ \ 1}{n \ \ r}}(K_n - r + 1, r)$. Assume that $K_n > 2r - 1$. It can be shown that $I_{\frac{m+r}{n}}(K_n - r + 1) < I_{\frac{m+r}{n}}(r, r)$ as a consequence of the property (38). Hence, we have

$$a_m < \frac{en}{m} \cdot I_{\frac{m \ \ 1}{n \ \ r}}(r, r) \leq \frac{en}{m} \cdot I_{\frac{m}{n \ \ r}}(r, r) \tag{50}$$

Assume that $\alpha = \alpha$ is the solution of the equation $I \ (r, r) = \frac{n \ \ r}{ne} \cdot \alpha$ in the range $0 < \alpha < \frac{1}{2}$. From Lemma 4.4, we know that this equation has only one solution in this range, and $I \ (r, r) \leq \frac{(n \ \ r)}{ne}$ for $0 < \alpha \leq \alpha$. Plugging in $\alpha = \frac{m}{n \ \ r}$, we have $I_{\frac{m}{n \ \ r}}(r, r) \leq \frac{m}{ne}$, which leads to $a_m < 1$ when $m \leq \lfloor \alpha \ (n - r) \rfloor$. Denoting $a := \max_m(a_m)$, we have

$$P_2 = \sum_{m = d\log(n)e}^{b \ \ (n \ \ r)c} (a_m)^m \leq \sum_{m = d\log(n)e}^{1} (a)^m \leq \frac{a^{\log(n)}}{1 - a} \tag{51}$$

where the geometric sum converges by virtue of $\lim_{n \ ! \ 1} a < 1$, leading to P_2 converging to zero as n gets large.

Now, similarly consider the third summation P_3.

$$P_3 \leq \sum_{m = d \ \ (n \ \ r)e}^{b \frac{n}{\ } c} \frac{n}{m} (P_{S_m})^m (P_{S^c_m})^{n \ \ m}$$

$$\leq \sum_{m = d \ \ n \ \ re}^{b \frac{n}{\ } c} \frac{n}{m} I_{\frac{m \ \ 1}{n \ \ r}}(K_n - r + 1, r)^{\frac{m}{n}}$$

$$\cdot \frac{n}{n - m} I_{\frac{n \ \ m \ \ r+1}{n \ \ r}}(K_n - r + 1, r)^{1 \ \ \frac{m}{n}} \#^{n}$$

Define

$$a_m := \frac{n}{m} I_{\frac{m \ \ 1}{n \ \ r}}(K_n - r + 1, r)^{\frac{m}{n}}$$

$$\cdot \frac{n}{n - m} I_{\frac{n \ \ m \ \ r+1}{n \ \ r}}(K_n - r + 1, r)^{1 \ \ \frac{m}{n}}$$

Again assuming that $K_n > 2r - 1$ and using the property (38), we have that $a_m < \frac{n}{m} I_{\frac{m \ \ 1}{n \ \ r}}(r, r)^{\frac{m}{n}} \cdot \frac{n}{n \ \ m} I_{\frac{n \ \ m \ \ r+1}{n \ \ r}}(r, r)^{1 \ \ \frac{m}{n}}$. Since both lower and upper limits of summation are a multiple of n, we can write $m = \alpha n$, for some α in the range $[\alpha \ , 1/2]$. Also using the property (37), we have

$$a_m < \frac{I_{\frac{m \ \ 1}{n \ \ r}}(r; r)}{\ } \cdot \frac{1 \ \ I_{\frac{m \ \ 1}{n \ \ r}}(r; r)}{1}^{1}$$

Again from Lemma 4.4, we know that $I \ (r, r) = c\alpha$ has only one solution for all α values in the range $0 < \alpha \leq 1/2$ when $0 < c \leq 1$, hence we can substitute $I \ (r, r) = c \ \alpha$ for some c in the range $0 < c \leq 1$ that satisfies $c = \frac{I \ (r; r)}{\ }$. Hence, to show $b \ := \frac{I_{\frac{m \ \ 1}{n \ \ r}}(r; r)}{\ } \cdot \frac{1 \ \ I_{\frac{m \ \ 1}{n \ \ r}}(r; r)}{1}^{1} \leq 1$, we can as well show $b_{c \ ;} := (c \ ) \ \frac{1 \ \ c}{1}^{1} \leq 1$, where $c = \frac{I \ (r; r)}{\ }$, $\alpha < \alpha \leq 1/2$. Since $c = \frac{1}{e}$ when $\alpha = \alpha$ and $c = 1$ when $\alpha = 1/2$, c will be in the range $[\frac{1}{e}, 1]$ when $\alpha < \alpha < 1/2$, hence we can as well consider all the range $\frac{1}{e} < c < 1$ and show the $b_{c;} := c \ \frac{1 \ \ c}{1}^{1} \leq 1$ where $\alpha < \alpha \leq 1/2$, $\frac{1}{e} < c \leq 1$. This is because this range $\frac{1}{e} < c \leq 1$ already includes c value that solves $c = \frac{I \ (r; r)}{\ }$. To show that $b_{c;} \leq 1$, when $\frac{1}{e} \leq c \leq 1$, $\alpha \leq \alpha \leq 1/2$, we need to check the boundary points and the stationary points of this surface. Starting with the derivative with respect to c, we have:

$$\frac{\partial(b_{c;} \ )}{\partial c} = \frac{\alpha}{c} \quad \frac{c - c\alpha}{1 - c\alpha} \qquad \frac{1 - c}{1 - \alpha} \tag{52}$$

In the range $0 < \alpha < 1/2$, $\frac{@(b_{c;} \ )}{@c} > 0$ when $\frac{1}{e} \leq c < 1$; and $\frac{@(b_{c;} \ )}{@c} = 0$ when $c = 1$, hence $c = 1$ gives the stationary points with respect to derivative over c. Now consider the derivative with respect to α.

$$\frac{\partial(b_{c;} \ )}{\partial \alpha} = c \quad \frac{1 - c\alpha}{1 - \alpha}^{1} \qquad \frac{1 - c}{1 - c\alpha} + \log \quad 1 - \frac{1 - c}{1 - c\alpha}$$

Since when $x \geq 0$, $x + \log(1 - x) = 0$ holds only when $x = 0$; $\frac{@}{@} = 0$ holds in the range $0 < \alpha < 1/2$, $\frac{1}{e} \leq c < 1$ only when $c = 1$. Further, since $c = 1$ gives the stationary points, we need to plug in $c = 1$ to $b_{c;}$ and find its maximum. We have $b_{1;} = 1 \ \frac{1}{1}^{1} = 1$. Also, considering the boundary points of $b_{c;}$, we can see that $b_{1;} = 1$, $b_{1;1=2} = 1$, $b_{\frac{1}{e};1=2} = \frac{1}{e}^{1=2} \ 2 - \frac{1}{e}^{1=2} < 1$ and $b_{\frac{1}{e};} = \frac{1}{e} \ \frac{1 \ \ =e}{1}^{1} < 1$ since $\frac{1}{e} < 1$ and $\frac{1 \ \ =e}{1} < 1$. Since $b_{c;} \leq 1$ at both stationary points and the boundary points, we can conclude that $b_{c;} \leq 1$, $\forall \ \frac{1}{e} \leq c < 1$, $\alpha \leq \alpha \leq 1/2$. This conclusion also leads to $b \leq 1$, $\forall \ \alpha \leq \alpha \leq 1/2$ and $a_m < 1$. Hence, we have:

$$P_3 \leq \sum_{m = d \ \ (n \ \ r)e}^{b \frac{n}{\ } c} (a_m)^n \leq \frac{n}{2}(\max_m(a_m))^n \tag{53}$$

where the sum converges by virtue of $\max_m(a_m) < 1$ since $\lim_{n \ ! \ 1} n(a)^n = 0$ when $0 < a < 1$, leading to P_3 converging to zero as n gets large. Since P_1, P_2 and P_3 all converge to zero as n gets large, $P_Z = P_1 + P_2 + P_3$ also converges to zero as n gets large. This concludes the proof of Theorem 3.5.

V. Conclusion

In this paper, we provide a comprehensive set of results on the r-robustness of the random K-out graph $\mathbb{H}(n; K_n)$, and the connectivity and giant component size of $\mathbb{H}(n; K_n, \gamma_n)$, i.e., random K-out graph with (randomly selected) γ_n nodes deleted. In addition to providing proofs of our results, we include computer simulations to validate our results in the finite node regime. To demonstrate the usefulness of the random K-out graphs, we compare our results on the random K-out graphs with results from ER graphs under similar settings, and determine that random K-out graphs attain r-robustness, connectivity or the occurrence of a giant component of a given size at a significantly lower mean node degree value compared to ER graphs. These results reinforce the usefulness of random K-out graphs in applications that require a certain degree of robustness or tolerance to nodes failing, being captured, or being dishonest; such as federated learning, consensus dynamics, distributed averaging and wireless sensor networks.