Random threshold graphs with exponential fitness: The width of the phase transition for connectivity

Armand M. Makowski
Department of Electrical and Computer Engineering, and Institute for Systems Research
University of Maryland, College Park, MD 20742.
Email: armand@isr.umd.edu

Osman Yağan
CyLab
Carnegie Mellon University, Pittsburgh, PA 15213.
Email: oyagan@andrew.cmu.edu

***Abstract*—We consider random threshold graphs where the fitness variables are exponentially distributed. Simulations show that the zero-one law for graph connectivity exhibits a sharp phase transition. We formalize this observation by providing exact asymptotics for the width of the phase transition in the many node regime.**

Keywords: Random threshold graphs, Exponential fitness, Connectivity, Phase transition.

I. Introduction

We are concerned with the following class of random graph models which have been proposed to describe some social networks: There are n nodes, labelled $k = 1, \ldots, n$, and to each node k we assign a *fitness* variable (or weight) ξ_k which measures its importance or rank. The random variables (rvs) $\xi_1, \ldots, \xi_n$ are assumed to form a collection of i.i.d. $\mathbb{R}$-valued rvs, each distributed according to some given probability distribution function $F : \mathbb{R} \to [0, 1]$.[1] For distinct $k, \ell = 1, \ldots, n$, we declare nodes k and ℓ to be adjacent if

$$\xi_k + \xi_\ell > \theta \tag{1}$$

for some θ in $\mathbb{R}$. We refer to the random graph defined by the adjacency notion (1) as a *random threshold* graph on the set of nodes $\{1, \ldots, n\}$, and hereafter we denote it by $\mathbb{T}(n; \theta)$.

These graphs are instances of hidden variable models and have been proposed as alternatives to the preferential attachment model of Barabási and Albert [2] to generate scale-free networks, e.g., see the papers [4], [5], [15] (and references therein). Random threshold graphs have recently been the focus of much activity; see the survey by Diaconis et al. [6] and the bibliography in [11]. For such random threshold graphs we are interested in the behavior of the property of graph connectivity. For convenience, for each θ in $\mathbb{R}$ we write

$$P(n; \theta) := \mathbb{P} [\, \mathbb{T}(n; \theta) \text{ is connected }] \tag{2}$$

with $n = 2, 3, \ldots$.

In particular, we seek to understand how these probabilities behave when the number n of nodes becomes large and the threshold value θ is scaled appropriately. This amounts to making θ depend on n by means of *scaling* functions $\theta : \mathbb{N}_0 \to \mathbb{R} : n \to \theta_n$, and to investigating the limit $\lim_{n\to\infty} P(n; \theta_n)$. We are particularly interested in conditions under which either

$$\lim_{n\to\infty} P(n; \theta_n) = 0 \tag{3}$$

or

$$\lim_{n\to\infty} P(n; \theta_n) = 1. \tag{4}$$

We naturally refer to the convergence statements (3) and (4) as a zero law and a one law, respectively.

Such zero-one laws have been discussed extensively in the context of other classes of random graphs, e.g., Erdös-Rényi graphs [3], [7], geometric random graphs [1], [13] and random key graphs [14], [16]. In a recent paper [11], the authors have considered the issue for random threshold graphs with *non-negative* fitness rvs. Distinguishing between *weak* and *strong* zero-one laws, we showed that the existence and type of a zero-one law, and the form of the critical scaling are completely determined by properties of F.

In this short conference paper we continue our investigation when F is an exponential distribution: For that special case a strong zero-one law is known to exist, and simulation results suggest that it exhibits a rather sharp transition. Our main result is a formalization of this observation, and takes the form of an exact asymptotics for the width of the phase transition. This is made possible by leveraging Theorem 3.2, an analog of the well-known double-exponential result (32) for graph connectivity in Erdős-Rényi graphs. Random threshold graphs and Erdős-Rényi graphs have very different connectivity structures, and so it is not surprising that the width of their phase transitions exhibits vastly different behavior, namely $\Theta(1)$ vs. $\Theta(n^{-1})$ in their natural parameters θ and α, respectively; see Section V for details. Yet, by reparametrizing random threshold graphs through their link assignment probability, we show that these different results can be somewhat harmonized.

All statements involving limits, including asymptotic equivalences, are always understood with n going to infinity. The rvs under consideration are all defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. All probabilistic statements are made with respect to this probability measure $\mathbb{P}$. The notation $\xrightarrow{P}_n$ (resp. $\Longrightarrow_n$) is used to signify convergence in probability (resp. convergence in distribution) with n going to infinity.

[1] What we call here a probability distribution function is also called a cumulative distribution function in other literatures.

II. Earlier work

In this section we summarize various results obtained in [11] concerning zero-one laws for connectivity under the non-negativity fitness assumption.

A. Assumptions

The setting of [11] is as follows: Let $\{\xi_k,\ k = 1, 2, \ldots\}$ denote a collection of i.i.d. $\mathbb{R}_+$-valued rvs defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ – We use ξ to refer to a generic representative for this sequence of i.i.d. rvs. The following assumptions are enforced on their common (cumulative) probability distribution function $F : \mathbb{R} \to [0, 1]$.

Assumption A: The probability distribution function $F : \mathbb{R} \to [0, 1]$

(i) *has support contained in* $\mathbb{R}_+$ *(so* $F(x) = 0$ *if* $x < 0$*);*
(ii) *is continuous on* $\mathbb{R}$*, or equivalently, has no atoms (and so cannot be degenerate at a single point, in particular, the origin);*

Assumption A-(ii) implies $F(0) = 0$, and is most easily satisfied by taking F to be absolutely continuous, say with density function $f : \mathbb{R} \to \mathbb{R}_+$ (so $f(x) = 0$ for $x < 0$ by Assumption A-(i)).

B. Weak vs. strong zero-one laws

Under Assumption A we have $P(n; \theta) = 1$ for all $n = 2, 3, \ldots$ whenever $\theta \leq 0$. Hence, in order to avoid uninteresting situations, we restrict the definition of a *scaling* to be any mapping $\theta : \mathbb{N}_0 \to \mathbb{R}_+$. Recall that we are interested in finding conditions on such scalings to ensure either $\lim_{n\to\infty} P(n; \theta_n) = 1$ or $\lim_{n\to\infty} P(n; \theta_n) = 0$. Typically there exist scalings, deemed *critical*, which act as boundary in the space of scalings between these two extremes. The terminology of McColm [12, p. 376], adapted here to the class of random threshold graphs, formalizes this idea in a number of different ways.

A *strong* zero-one law is said to hold with *critical* scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ if for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying

$$\lim_{n\to\infty} \frac{\theta_n}{\theta^\star_n} = c \tag{5}$$

for some $c > 0$, we have

$$\lim_{n\to\infty} P(n; \theta_n) = \begin{cases} 1 & \text{if } 0 < c < 1 \\ \\ 0 & \text{if } 1 < c. \end{cases} \tag{6}$$

Any scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ appearing in (5)-(6) is called a *strong* critical scaling.

On the other hand, a *weak* zero-one law is said to hold with *critical* scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ if for scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ we have

$$\lim_{n\to\infty} P(n; \theta_n) = \begin{cases} 1 & \text{if } \lim_{n\to\infty} \frac{\theta_n}{\theta^\star_n} = 0 \\ \\ 0 & \text{if } \lim_{n\to\infty} \frac{\theta_n}{\theta^\star_n} = \infty. \end{cases} \tag{7}$$

Any scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ appearing in (7) is called a *weak* critical scaling.

C. Characterizing zero-one laws

For each $n = 2, 3, \ldots$, the probability (2) can be expressed in terms of two order statistics associated with the underlying rvs $\xi_1, \ldots, \xi_n$, namely their maxima and minima given by

$$M_n := \max(\xi_1, \ldots, \xi_n) \tag{8}$$

and

$$M^\star_n := \min(\xi_1, \ldots, \xi_n), \tag{9}$$

respectively. In [11] we established the following representation for the probability of graph connectivity.

Proposition 2.1: Under Assumption A on the probability distribution function F*, we have*

$$P(n; \theta) = \mathbb{P}[M^\star_n + M_n > \theta], \quad \theta > 0 \tag{10}$$

for each $n = 2, 3, \ldots$.

With the help of (10) it is easy to give a complete characterization of strong zero-one laws [11].

Theorem 2.2: Assumption A is enforced on the probability distribution function F*. If* $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ *is a scaling which satisfies*

$$\lim_{n\to\infty} \theta^\star_n = \infty, \tag{11}$$

then the strong zero-one law (6) holds for graph connectivity with critical scaling $\theta^\star : \mathbb{N}_0 \to (0, \infty)$ *if and only if*

$$\frac{M_n}{\theta^\star_n} \xrightarrow{P}_n 1. \tag{12}$$

A characterization of weak zero-one laws is also available but will not be used here; see [11] for details.

III. Exponential fitness

From now on we focus on the special case when ξ is exponentially distributed with parameter $\lambda > 0$, i.e.,

$$\mathbb{P}[\xi \leq x] = 1 - e^{-\lambda x}, \quad x \geq 0. \tag{13}$$

This case was considered in [5], [15] to show that even non-scale free distributions can generate scale-free networks.

A. Classical limiting results

Of particular interest for what follows is the Gumbel distribution $G : \mathbb{R} \to \mathbb{R}_+$ given by

$$G(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}. \tag{14}$$

Any rv Λ distributed according to the Gumbel distribution is called a Gumbel rv, i.e.,

$$\mathbb{P}[\Lambda \leq x] = G(x), \quad x \in \mathbb{R}. \tag{15}$$

The following limiting results from Extreme Value Theory are well known [8, Example 3.2.7, p. 125].

Proposition 3.1: We have

$$\frac{\lambda M_n}{\log n} \xrightarrow{P}_n 1 \tag{16}$$

and

$$\lambda M_n - \log n \Longrightarrow_n \Lambda. \tag{17}$$

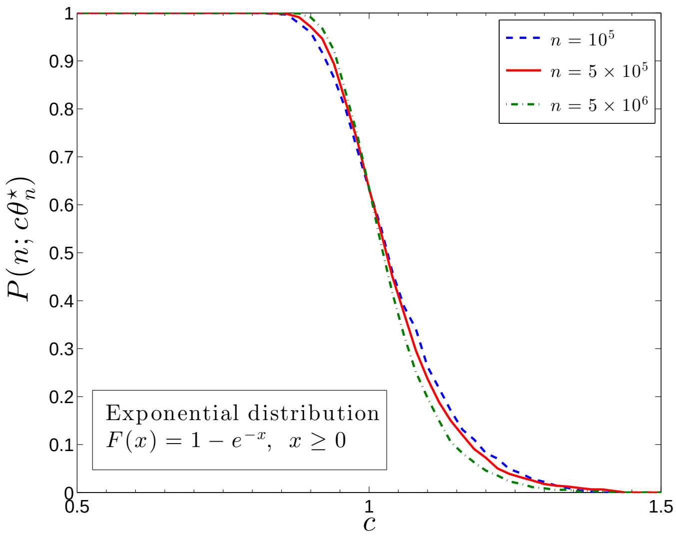

Fig. 1. *Exponential distribution (13) with* $\lambda = 1$.

Note that (16) is an easy consequence of (17), and this last convergence can be obtained by elementary arguments.

Theorem 2.2 can now be applied in the exponential case (13): From (16) we conclude that a strong zero-one law exists with critical scaling $\theta^\star_n : \mathbb{N}_0 \to (0, \infty)$ given by

$$\theta^\star_n = \lambda^{-1} \log n, \quad n = 1, 2, \ldots \tag{18}$$

Thus, for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ such that

$$\theta_n \sim c \cdot \frac{\log n}{\lambda}$$

for some $c > 0$, we have $\lim_{n\to\infty} P(n; \theta_n) = 1$ (resp. $\lim_{n\to\infty} P(n; \theta_n) = 0$) if $0 < c < 1$ (resp. $1 < c$). This state of affairs is confirmed through the simulation results displayed in Figure 1 where $\lambda = 1$ so that the critical scaling (18) is now $\theta^\star_n = \log n$ for each $n = 1, 2, \ldots$. With $c > 0$ we estimate $P(n; c\theta^\star_n)$ by the *empirical probability* that the random threshold graph is connected under the scaling $n \to c\theta^\star_n$; this quantity is obtained by averaging over 5000 independent realizations.

B. Sharper results for the exponential distribution

As should be apparent already from Figure 1, this strong zero-one law exhibits a rather sharp transition. The remainder of the paper is devoted to formalizing this observation. Throughout we write $x^+ = \max(0, x)$ for any scalar x in $\mathbb{R}$.

Write any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ in the form

$$\theta_n = \lambda^{-1} (\log n + \gamma_n)^+, \quad n = 1, 2, \ldots \tag{19}$$

for some sequence $\gamma : \mathbb{N}_0 \to \mathbb{R}$ – There is no loss of generality in doing so. In [11] we established the following result.

Theorem 3.2: If the distribution function $F : \mathbb{R} \to [0, 1]$ *is the exponential distribution (13) with parameter* $\lambda > 0$*, then*

$$\lim_{n\to\infty} P\left(n; \lambda^{-1} (\log n + \gamma_n)^+\right) = 1 - e^{-e^{-\Gamma}} \tag{20}$$

for any scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ *written in the form (19) with sequence* $\gamma : \mathbb{N}_0 \to \mathbb{R}$ *satisfying*

$$\lim_{n\to\infty} \gamma_n = \Gamma \tag{21}$$

for some Γ *in* $\mathbb{R}$.

The proof given in [11] can be easily summarized: For a scaling $\theta : \mathbb{N}_0 \to \mathbb{R}_+$ put in the form (19), use the expression (10) to evaluate $P(n; \theta_n)$ for each $n = 2, 3, \ldots$. Under (21) the conclusion is readily obtained by combining (17) with the fact that $\lim_{n\to\infty} M^\star_n = 0$ a.s.

Theorem 3.2 is in the spirit of the celebrated double-exponential result for graph connectivity in Erdős-Rényi graphs; see discussion in Section V. In particular, if in (21) we take $\gamma_n \equiv \Gamma$ for all $n = 1, 2, \ldots$, then (20) becomes

$$\lim_{n\to\infty} P\left(n; \lambda^{-1} (\log n + \Gamma)^+\right) = 1 - e^{-e^{-\Gamma}}. \tag{22}$$

IV. Width of the Phase Transition

Fix $n = 3, 4, \ldots$. the mapping $\theta \to P(n; \theta)$ is easily seen to be continuous and strictly decreasing on $\mathbb{R}_+$. Thus, for each p in $(0, 1)$, there exists a unique solution to the equation

$$P(n; \theta) = p, \quad \theta > 0.$$

With $\theta_n(p)$ denoting this unique solution, we obviously have

$$P(n; \theta_n(p)) = p. \tag{23}$$

Note also that there exists a unique scalar Γ in $\mathbb{R}$ such that

$$1 - e^{-e^{-\Gamma}} = p.$$

This unique scalar, denoted by $\Gamma(p)$, is given by

$$\Gamma(p) = -\log\left(-\log(1 - p)\right).$$

The next result constitutes the main technical contribution of the paper, and provides an explicit approximation to $\theta_n(p)$ for large n.

Theorem 4.1: For each p *in* $(0, 1)$ *we have*

$$\theta_n(p) = \lambda^{-1} (\log n + \Gamma(p)) + o(1). \tag{24}$$

Proof. Pick p in $(0, 1)$. We select $\varepsilon > 0$ and $\eta > 0$ so that the conditions

$$\eta + \varepsilon < \min(p, 1 - p) \quad \text{and} \quad 3\eta < \varepsilon \tag{25}$$

are both satisfied. This ensures that the three intervals $(p - \varepsilon - \eta, p - \varepsilon + \eta)$, $(p - \eta, p + \eta)$, and $(p + \varepsilon - \eta, p + \varepsilon + \eta)$ are non-overlapping intervals, each contained in $(0, 1)$.

By virtue of (22) (with $\Gamma = \Gamma(p \pm \varepsilon)$), there exists a positive integer $n(\varepsilon, \eta)$ such that the conditions

$$\log n + \Gamma(p \pm \varepsilon) \geq 0 \tag{26}$$

and

$$\left| P\left(n; \lambda^{-1} (\log n + \Gamma(p \pm \varepsilon))^+\right) - (p \pm \varepsilon) \right| < \frac{\eta}{2} \tag{27}$$

all hold whenever $n \geq n(\varepsilon, \eta)$. On this range it follows from (26) and (27) that both inequalities

$$P\left(n; \lambda^{-1}\left(\log n + \Gamma(p-\varepsilon)\right)\right) < (p-\varepsilon) + \frac{\eta}{2}$$

and

$$(p+\varepsilon) - \frac{\eta}{2} < P\left(n; \lambda^{-1}\left(\log n + \Gamma(p+\varepsilon)\right)\right)$$

hold. Making use of (23) and of the non-overlapping condition (25) we conclude that

$$P\left(n; \lambda^{-1}\left(\log n + \Gamma(p-\varepsilon)\right)\right) < P\left(n; \theta_n(p)\right)$$

and

$$P\left(n; \theta_n(p)\right) < P\left(n; \lambda^{-1}\left(\log n + \Gamma(p+\varepsilon)\right)\right).$$

Since the mapping $\theta \to P(n; \theta)$ is decreasing, it follows that

$$\theta_n(p) < \lambda^{-1}\left(\log n + \Gamma(p-\varepsilon)\right)$$

and

$$\lambda^{-1}\left(\log n + \Gamma(p+\varepsilon)\right) < \theta_n(p).$$

Rearranging terms in these inequalities and then letting n go to infinity we find

$$\limsup_{n\to\infty}\left(\theta_n(p) - \lambda^{-1}\log n\right) \leq \lambda^{-1}\Gamma(p-\varepsilon)$$

and

$$\lambda^{-1}\Gamma(p+\varepsilon) \leq \liminf_{n\to\infty}\left(\theta_n(p) - \lambda^{-1}\log n\right),$$

inequalities whose left and right handsides, respectively, do not depend on ε. Let ε go to zero in the last two inequalities and observe that

$$\lim_{\varepsilon\downarrow 0}\Gamma(p+\varepsilon) = \lim_{\varepsilon\downarrow 0}\Gamma(p-\varepsilon) = \Gamma(p).$$

It is now plain that

$$\lim_{n\to\infty}\left(\theta_n(p) - \lambda^{-1}\log n\right) = \lambda^{-1}\Gamma(p),$$

or equivalently,

$$\theta_n(p) - \lambda^{-1}\log n = \lambda^{-1}\Gamma(p) + o(1).$$

The conclusion (24) follows. ■

The phase transition already apparent in Figure 1 has a width which we characterize through the quantity

$$\delta_n(p) = \theta_n(1-p) - \theta_n(p), \qquad \begin{array}{c} n = 3, 4, \ldots \\ p \in [0, \frac{1}{2}). \end{array} \tag{28}$$

Note that p in $[0, \frac{1}{2})$ is equivalent to $p < 1-p$, so that $\theta_n(1-p) < \theta_n(p)$, hence $\delta_n(p) < 0$. The transition width $\delta_n(p)$ measures the decrease in the threshold value θ needed in the n node network to drive the probability of connectivity from level p to level $1-p$. The more slowly $|\delta_n(p)|$ grows as a function of n, the sharper the phase transition. This rate of decay is available as a direct consequence of Theorem 4.1.

Corollary 4.2: For each p *in* $(0, \frac{1}{2})$ *we have*

$$\delta_n(p) = -\lambda^{-1}C(p) + o(1) \tag{29}$$

where

$$C(p) = \log\left(\frac{\log p}{\log(1-p)}\right) > 0. \tag{30}$$

V. Discussion

A. Phase transition in Erdős-Rényi graphs

Results similar to Theorem 4.1 and Corollary 4.2 are available for other classes of random graphs, e.g., Erdős-Rényi graphs (see below) and geometric random graphs (as discussed in [10] in one dimension).

For instance, with α in $[0,1]$ and $n = 2, 3, \ldots$, consider the Erdős-Rényi graph $\mathbb{G}(n;\alpha)$ on the vertex set $\{1, \ldots, n\}$ with link assignment probability α. Let $P_{\rm ER}(n;\alpha)$ denote the probability that $\mathbb{G}(n;\alpha)$ is connected. The mapping $\alpha \to P_{\rm ER}(n;\alpha)$ is continuous and strictly increasing on $[0,1]$. Given p in $(0,1)$, there exists a unique solution $\alpha_n(p)$ to the equation

$$P_{\rm ER}(n;\alpha) = p, \quad \alpha \in (0,1).$$

Its behavior for large n is given by

$$\alpha_n(p) = \frac{\log n + \Gamma(1-p)}{n} + o\left(n^{-1}\right). \tag{31}$$

This a consequence of the double-exponential result for graph connectivity in Erdős-Rényi graphs [3, Thm. 7.3, p. 164], [7, Thm. 3.10, p. 42], namely

$$\lim_{n\to\infty} P_{\rm ER}\left(n; \left(\frac{\log n + c_n}{n}\right)^+\right) = e^{-e^{-c}} \tag{32}$$

whenever the sequence $c : \mathbb{N}_0 \to \mathbb{R}$ satisfies $\lim_{n\to\infty} c_n = c$ for some c in $\mathbb{R}$ – Contrast this with (20)-(21). The arguments leading to (31) are similar to the ones used in [9], [10] for one-dimensional geometric random graphs. This time, in analogy with (28), we need to consider

$$\delta_{{\rm ER},n}(p) = \alpha_n(1-p) - \alpha_n(p), \qquad \begin{array}{c} n = 3, 4, \ldots \\ p \in [0, \frac{1}{2}) \end{array}$$

so that

$$\delta_{{\rm ER},n}(p) = \frac{C(p)}{n} + o\left(n^{-1}\right) \tag{33}$$

by virtue of (31) with $C(p)$ given by (30). Details are left to the interested reader,

B. Reparametrization

Connectivity in both random threshold graphs and Erdős-Rényi graphs exhibit a phase transition, the width of which is given by (29) and (33), respectively. The asymptotic behaviors are quite different: While $\lim_{n\to\infty} \delta_{{\rm ER},n}(p) = 0$, we note that $\lim_{n\to\infty} \delta_n(p) = -\lambda^{-1}C(p) < 0$ on the range $(0, \frac{1}{2})$. This suggests that for large n, the probability $P(n;\theta)$ (as a function of θ) decreases from $1-p$ to p somewhat *linearly* over a nearly constant range of positive size $\lambda^{-1}C(p)$ and with negative slope

$$\frac{(1-p)-p}{-\lambda^{-1}C(p)} = -\frac{\lambda(1-2p)}{C(p)}.$$

On the other hand, for large n, the probability $P_{\mathrm{ER}}(n;\alpha)$ (as a function of α) jumps in a *step-like* manner (with infinite slope) from p to $1-p$.

One might possibly argue that the observed difference in behavior is misleading because θ lives in the unbounded half-line $(0,\infty)$ while α ranges over the compact interval $(0,1)$. Therefore, one approach to render the results comparable would be to reparametrize random threshold graphs in terms of their probability of link assignment. To that end, we define

$$\beta(\theta) = \mathbb{P}\left[\xi_1 + \xi_2 > \theta\right], \quad \theta > 0.$$

For each $n = 2, 3, \ldots$, as the probability of link assignment in $\mathbb{T}(n;\theta)$, the parameter $\beta(\theta)$ is akin to the link assignment probability α in the Erdős-Rényi graph $\mathbb{G}(n;\alpha)$. Under the exponential distribution (13), we get

$$\beta(\theta) = h(\lambda\theta)$$

where $h : \mathbb{R}_+ \to [0,1]$ is the continuous and strictly decreasing mapping given by

$$h(x) = (1+x)\, e^{-x}, \quad x \geq 0.$$

Since θ and $\beta(\theta)$ are in one-to-one correspondence with each other, we can think of $\mathbb{T}(n;\theta)$ as $\mathbb{T}(n;\beta)$ where $\beta = h(\lambda\theta)$

Note that $\beta_n(p) = h(\lambda\theta_n(p))$ is simply the probability of link assignment in the n node random threshold graph that ensures connectivity with probablity p; of course its counterpart in Erdős-Rényi graphs is $\alpha_n(p)$. We easily check that

$$\begin{aligned}\beta_n(p) &= (1 + \log n + \Gamma(p) + o(1))\, e^{-(\log n + \Gamma(p) + o(1))} \\ &= \left(\frac{\log n + \Gamma(p)}{n} + \Theta(n^{-1})\right) e^{-\Gamma(p)} e^{o(1)}\end{aligned}$$

whence

$$\beta_n(p) \sim \left(\frac{\log n + \Gamma(p)}{n} + \Theta(n^{-1})\right) \cdot \log\left(\frac{1}{1-p}\right). \quad (34)$$

While the similarity of (34) with (31) is encouraging, a finer asymptotic analysis is required to conclude that the transition width $\beta_n(1-p) - \beta_n(p)$ (expressed in the β parameter) behaves like (33).

C. Normalization to the critical scaling

Another possibility to harmonize the two sets of results (29) and (33) is to normalize the transition width to the critical scaling. For simplicity consider the case when p is in the interval $[0, \frac{1}{2})$. For random threshold graphs this idea leads to

$$\frac{\delta_n(p)}{\theta_n^\star} = \frac{-\lambda^{-1}C(p) + o(1)}{\lambda^{-1}\log n} = \frac{-C(p) + o(1)}{\log n}.$$

For Erdős-Rényi graphs, the critical scaling $\alpha^\star : \mathbb{N}_0 \to [0,1]$ for graph connectivity is given by

$$\alpha_n^\star = \frac{\log n}{n}, \quad n = 1, 2, \ldots$$

so that

$$\begin{aligned}\frac{\delta_{\mathrm{ER},n}(p)}{\alpha_n^\star} &= \frac{\frac{C(p)}{n} + o(n^{-1})}{\frac{\log n}{n}} \\ &= \frac{C(p) + o(1)}{\log n}.\end{aligned}$$

Thus, in spite of very different connectivity structures, we have

$$\left|\frac{\delta_n(p)}{\theta_n^\star}\right| \sim \frac{\delta_{\mathrm{ER},n}(p)}{\alpha_n^\star}.$$

One may wonder wether the absolute value of this ratio indeed provides some form of invariant across many classes of random graphs. To be continued!

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REFERENCES

[1] M.J.B. Appel and R.P. Russo, "The connectivity of a graph on uniform points on $[0,1]^d$," *Statistics & Probability Letters* **60** (2002), pp. 351-357.

[2] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science* **286** (1999), pp. 509-512.

[3] B. Bollobás, *Random Graphs*, Second Edition, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (UK), 2001.

[4] M. Boguñá and R. Pastor-Satorras, "Class of correlated random networks with hidden variables," *Physical Review* **E 68** (2003), 036112.

[5] G. Caldarelli, A. Capocci, P. De Los Rios and M.A. Muñoz, "Scale-free networks from varying vertex intrinsic fitness," *Physical Review Letters* **89** (2002), 258702.

[6] P. Diaconis, S. Holmes and S. Janson, "Threshold graph limits and random threshold graphs," *Internet Mathematics* **5** (2008), pp. 267-320

[7] M. Draief and L. Massoulié, *Epidemics and Rumours in Complex Networks*, London Mathematical Society Lecture Note Series **369**, Cambridge University Press, Cambridge (UK), 2010.

[8] P. Embrechts, C. Klüppelberg and Th. Mikosch, *Modelling Extremal Events*, Applications of Mathematics: Stochastic Modelling and Applied Probability **33**, Springer-Verlag, Berlin Heidelberg, 1997.

[9] G. Han and A.M. Makowski, "Very sharp thresholds in one-dimensional MANETs," in Proceedings of the 2006 IEEE International Conference on Communications (ICC 2006), Istanbul (Turkey), June 2006.

[10] G. Han and A.M. Makowski, "Sensitivity of critical transmission ranges to node placement distributions," *IEEE Journal on Selected Areas in Communications* **JSAC–27** (2009), Special Issue on Stochastic Geometry and Random Graphs for Wireless Networks, pp. 1066-1078.

[11] A. M. Makowski and O. Yağan, "Scaling laws for connectivity in random threshold graph models with non-negative fitness variables," *IEEE Journal on Selected Areas in Communications* **JSAC–31** (2013), Special Issues on Emerging Technologies in Communications (Area 4: Social Networks). Available at http://www.andrew.cmu.edu/user/oyagan/.

[12] G.L. McColm, "Threshold functions for random graphs on a line segment," *Combinatorics, Probability and Computing* **13** (2004), pp. 373-387.

[13] M.D. Penrose, *Random Geometric Graphs*, Oxford Studies in Probability **5**, Oxford University Press, New York (NY), 2003.

[14] K. Rybarczyk, "Diameter of the uniform random intersection graph with a note on the connectivity and the phase transition," *Discrete Mathematics* **311** (2011), pp. 1998-2019.

[15] V.D.P. Servedio and G. Caldarelli, "Vertex intrinsic fitness: How to produce arbitrary scale-free networks," *Physical Review E* **70** (2004), 056126.

[16] O. Yağan and A.M. Makowski, "Zero-one laws for connectivity in random key graphs." *IEEE Transactions on Information Theory* **IT-58** (2012), pp. 2983-2999.