Legendre polynomial expansion for analysis of linear one-dimensional inhomogeneous optical structures and photonic crystals

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A Legendre polynomial expansion of electromagnetic fields for analysis of layers with an inhomogeneous refractive index profile is reported. The solution of Maxwell's equations subject to boundary conditions is sought in a complete space spanned by Legendre polynomials. Also, the permittivity profile is interpolated by polynomials. Different cases including computation of reflection–transmission coefficients of inhomogeneous layers, band-structure extraction of one-dimensional photonic crystals whose unit-cell refractive index profiles are inhomogeneous, and inhomogeneous planar waveguide analysis are investigated. The presented approach can be used to obtain the transfer matrix of an arbitrary inhomogeneous monolayer holistically, and approximation of the refractive index or permittivity profile by dividing into homogeneous sublayers is not needed. Comparisons with other well-known methods such as the transfer-matrix method, WKB, and effective index method are made. The presented approach, based on a nonharmonic expansion, is efficient, shows fast convergence, is versatile, and can be easily and systematically employed to analyze different inhomogeneous structures. © 2006 Optical Society of America

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1. INTRODUCTION

Recent advances in fabrication technologies have brought about the possibility of having materials with arbitrary refractive index profiles made through precise computer control of the fabrication parameters.[1] Many materials used in optical devices are composed of silicon compounds, benefiting from the low cost of silicon technology and the well-developed techniques used in microelectronics. Many optical devices such as waveguides and inhomogeneous photonic crystals can be made using these materials with arbitrary refractive index profiles. Tailoring the refractive index profiles to meet certain desired properties in such structures is of great importance.[2–4] Therefore investigating different optical phenomena such as scattering, guided-mode analysis, and band-structure calculation in these structures is necessary and requires well-established, accurate, and efficient methods. Reflection and transmission calculations from inhomogeneous media are also of great importance for characterization of fabricated devices.[5]

Lightwave propagation in periodic structures has been the subject of considerable attention lately.[6] By optimization of the refractive index profile of the unit cell of photonic crystals, their band structure can be engineered. Different methods have been proposed for calculating band structures since Brillouin[7] made the earliest attempt at investigating the wave propagation in periodic media, such as the plane-wave expansion method[8] (PWE), the transfer-matrix method[9] (TMM), and numerical approaches such as the finite-difference time domain[10] and finite-element method (FEM).[11] The PWE is based on the Fourier expansion of the periodic permittivity and the amplitude of Bloch waves and suffers from poor convergence especially in those cases in which the refractive index has a harsh profile. The TMM requires a staircase approximation of the refractive index profile; thus accurate calculation of high-order bands requires finer approximation of the refractive index profile. Numerical methods are often computationally intensive and yield less physical intuition and thus are not efficient as optimization and synthesis tools.

Although exact solutions of the wave equation for inhomogeneous waveguides of some specific index profiles are available in closed form,[12] many different approximate methods have been developed for analyzing inhomogeneous waveguides of arbitrary refractive index profiles. The WKB method[13,14] is widely used as an approximate approach for analyzing inhomogeneous waveguides, though its results show reasonable accuracy only when index variation is slow and the transverse waveguide dimension is much larger than the wavelength at free space.[15] Furthermore, the presence of turning points, causing failure in calculating field profiles, has motivated a considerable effort in the literature for modifying the WKB method.[15–17] Also, in the case of truncated index profiles, large errors are observed near the mode cutoff frequencies; such large errors can be ameliorated by one's contriving new schemes for correcting the usually approximated phase shift of $\pi/2$ in conventional WKB.[15] Another well-known approach is the TMM,[18,19] which yields exact solutions for step-index waveguides. However, this approach, requiring a staircase approximation of the refractive index profile for inhomogeneous waveguides, becomes tedious and results in complicated transcendental dispersion equations, which cannot be easily solved. Other approaches such as the variational method,[20] perturbation method,[21] differential TMM,[22–24] and numerical methods such as the beam propagation method,[25] FEM,[26] and others[27,28] can be used for analyzing inhomogeneous waveguides.

In the variational approach, the choice of the trial field is critical, since it should closely resemble the exact solution. The perturbation method is applicable only in those cases in which a closely related problem having an exact solution exists. The differential TMM yields approximate analytic solutions.[22–24] Even though a modified version of this approach, improving the accuracy of the obtained results, has been proposed,[29] it is still problematic for those cases in which turning points are encountered. The aforementioned numerical methods for analyzing inhomogeneous waveguides, despite their accuracy, are usually computationally extensive and cannot intuitively yield the physical insight behind the problem. Most of the previously listed methods give transcendental dispersion equations, which are difficult to solve. In this regard, different mathematical approaches are put forth for finding the zeros of the obtained dispersion equation.[30,31] A method based on polynomial expansion for extraction of electromagnetic eigenmodes in layered structures has been presented.[32] It has been shown that, with this method, algebraic and easy-to-solve dispersion equations can be derived for analysis of stratified waveguides. Similar polynomial expansion has also been applied to analysis of diffraction gratings.[33] Here, an extension of such polynomial expansion-based methods, though with a different formulation for analysis of inhomogeneous layers, is presented. The presented method is based on a Legendre polynomial expansion of electromagnetic fields and is proposed for analyzing inhomogeneous layers. The permittivity profile in the inhomogeneous medium is interpolated in terms of polynomials, and then this polynomial-based interpolation is absorbed in the Legendre expansion of electromagnetic fields in a wave equation by one's employing an interesting property of the Legendre polynomials. Thus, not only are the electromagnetic fields expanded in the complete space spanned by Legendre polynomials but also the permittivity is expressed in terms of polynomials. This is in contrast to many conventional methods such as the PWE that rely on the Fourier expansion of the permittivity function and thus avoids the Gibbs phenomenon in analyzing periodic structures. By following this method for analysis of inhomogeneous waveguides, one can derive well-behaved algebraic dispersion equations instead of complicated transcendental dispersion equations. These algebraic dispersion equations can be easily solved. Also, using the proposed method, one can obtain analytic expressions describing the dependency of the optical response of the structures on their constituting parameters such as the permittivity profile. In this regard an example is given to analytically demonstrate how the range of single-mode operation of parabolic waveguides can be engineered by one's changing the permittivity profile. Such analytic expressions not only yield physical insight into the problem but also can be useful for finding rules of thumb in synthesis and optimization processes. The proposed method is efficient in many commonly encountered problems and can be easily-implemented for analyzing different problems including scattering of light waves from inhomogeneous layers, inhomogeneous photonic-crystal band-structure calculation, and guided-mode analysis of inhomogeneous waveguides. The formulation is given so that it can be implemented easily and in an automated manner. This paper is organized as follows. A polynomial expansion formulation for analysis of inhomogeneous layers is presented in a general case in Section 2. In this section the appropriate boundary conditions for calculation of reflection–transmission coefficients, band-structure calculation of periodic structures, and guided-mode analysis are discussed distinctly. Examples and results are given in Section 3, and, finally, conclusions are made in Section 4.

2. FORMULATION

In accordance with the structure shown in Fig. 1, Maxwell's equations in a source-free inhomogeneous medium can be combined to obtain the Helmholtz equation for TE polarization as

$$\nabla^2 E_y(x,z) + k^2 n^2(x) E_y(x,z) = 0, \tag{1}$$

where k is the free-space wave vector and $n(x)$ is the relative refractive index profile.

For any refractive index profile, this equation governs the electromagnetic fields that can propagate in the structure. As a result of continuous translational symmetry in the z direction, the general form of the solution of Eq. (1) is given by

$$E_y(x,z) = U(x)\exp(-j\beta z), \tag{2}$$

where β is the propagation constant determined from the phase-matching condition.

Substituting this form of solution in Eq. (1) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}U(x) + [k^2 n^2(x) - \beta^2]U(x) = 0. \tag{3}$$

The above equation, subject to specific boundary conditions at $x=0$ and $x=d$, has to be solved to determine $U(x)$. This second-order differential equation has analytic solu-

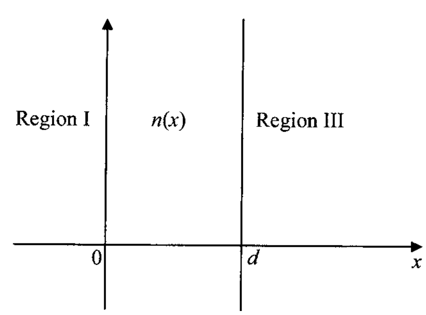

Fig. 1. One-dimensional inhomogeneous medium.

tions for some specific functions of $n(x)$[12]; however, there are many practical profiles for which closed-form solutions are not available, and thus a considerable amount of effort is spent on developing methods for analyzing such structures. In the presented method the solution is sought in a complete space spanned by Legendre polynomials. Thus, $U(x)$ is expanded as

$$U(x)=\sum_{m=0}^{+\infty} q_m P_m(\xi), \tag{4}$$

where $P_m(\xi)$ are Legendre polynomials, q_m are unknown expansion coefficients that are determined later, d is the layer thickness, and $\xi=(2x-d)/d$ is an auxiliary variable that maps $0<x<d$ to $-1<\xi<1$. This kind of mapping facilitates future calculations involving Legendre polynomials. Note that, in practice, the expansion in Eq. (4) is inevitably truncated to a finite number of terms, say, M, determined by the required level of accuracy.

After substituting the expansion of Eq. (4) in Eq. (3), one obtains

$$\sum_{m=0}^{+\infty} q_m \frac{\mathrm{d}^2}{\mathrm{d}x^2} P_m(\xi)+[k^2n^2(\xi)-\beta^2]\sum_{m=0}^{+\infty} q_m P_m(\xi)=0. \tag{5}$$

An interesting recursive property of the Legendre polynomials calls for

$$xP_m(x)=\frac{m+1}{2m+1}P_{m+1}(x)+\frac{m}{2m+1}P_{m-1}(x). \tag{6}$$

That is, any power of x can be easily absorbed in the Legendre polynomials. Thus, it can be readily shown that

$$\sum_{m=0}^{+\infty} xq_m P_m(x)=\sum_{m=0}^{+\infty} \chi_m P_m(\xi), \tag{7}$$

where

$$\chi_m=\frac{m}{2m-1}q_{m-1}+\frac{m+1}{2m+3}q_{m+1}. \tag{8}$$

Equation (8) can be arranged in a matrix format systematically as

$$[\bar{\chi}_m]=[\chi][\bar{q}_m]. \tag{9}$$

Therefore, once the matrix $[\chi]$ is generated, any power of x can be easily absorbed in the expansion given in Eq. (7), by one's multiplying the corresponding power of $[\chi]$ to the vector $[q_m]$.

The permittivity profile, $n^2(\xi)$, in Eq. (5) can be interpolated as

$$n^2(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3+\cdots+a_h\xi^h, \tag{10}$$

where a_i are expansion coefficients determined by fitting the above expansion to the profile of $n^2(\xi)$ by using such algorithms as least-squares error. The appropriate value of h is chosen so that the expansion given in Eq. (10) becomes an acceptable fit to the profile of $n^2(\xi)$. In many cases such as linear and parabolic profiles, $n^2(\xi)$ can be exactly described by maintaining a finite h, whereas profiles such as exponential, Gaussian, and other more complicated ones can be described within an acceptable approximation by using a proper value of h. Notwithstanding, one can maintain any required number of terms in Eq. (10) to reach the desired level of accuracy in the polynomial interpolation of $n^2(\xi)$.

By substituting for $n^2(\xi)$ from Eq. (10), doing some mathematical manipulations, and projecting Eq. (3) on Legendre polynomial basis functions, one can obtain a set of equations as

$$[\bar{r}_m]+k^2\left(\frac{d}{2}\right)^2\sum_n a_n[\chi]^n[\bar{q}_m]-\beta^2\left(\frac{d}{2}\right)^2[\bar{q}_m]=0, \tag{11}$$

where m runs from 0 to $M-2$ and r_m are the expansion coefficients of the second derivative in terms of q_m, defined by

$$r_m=\frac{2m+1}{2}\sum_{\substack{l=m+2t\\ t=1,2,3,\ldots,}}^{M_i}(l+m+1)(l-m)q_l. \tag{12}$$

Note that, thus far, the number of unknowns is $M+1$, since in the truncated version of the expansion of Eq. (4) m runs from 0 to M, resulting in $M+1$ q_m unknown coefficients. Equation (11) constitutes a set of $M-1$ independent equations resulting from the Helmholtz equation, and two other equations arising from boundary conditions at $x=0$ and $x=d$ are needed to fully determine the unknown coefficients.

Boundary conditions call for the continuity of the tangential electromagnetic fields at $x=0$ and $x=d$. The electromagnetic fields in region I at $x=0$ can be described as

$$E_y=U_1\exp(-j\beta z),$$

$$H_z=\frac{1}{\omega\mu}V_1\exp(-j\beta z) \tag{13}$$

and in region III at $x=d$ as

$$E_y=U_3\exp(-j\beta z),$$

$$H_z=\frac{1}{\omega\mu}V_3\exp(-j\beta z). \tag{14}$$

In these expressions, ω is the angular frequency, and μ is the magnetic permeability that is equal to the free-space permeability, since here only nonmagnetic media are dealt with. The factor $1/\omega\mu$ is included for simplification of subsequent formulas. Continuity of tangential electric and tangential magnetic fields requires

$$U_1=\sum_{m=0}^{M} q_m P_m(-1),$$

$$V_1=j\sum_{m=0}^{M} q_m P'_m(-1) \tag{15}$$

at $x=0$, and

$$U_3 = \sum_{m=0}^{M} q_m P_m(1),$$

$$V_3 = j \sum_{m=0}^{M} q_m P'_m(1) \quad (16)$$

at $x=d$. In the above equations $P'_m(\xi)|_{\xi=\pm 1}$ is the first derivative of $P_m(\xi)$ evaluated at $\xi=\pm 1$. Equations (11), (15), and (16) constitute a complete set of equations for determining unknown q_m coefficients. Equations (15) and (16) each constitute two equations that can be combined to yield one equation describing the boundary condition at $x=0$ and one equation at $x=d$. Although boundary equations are generally described here as Eqs. (15) and (16), the way that U_1 and V_1 and U_3 and V_3 are related to each other depends on the nature of the problem. Henceforth, three distinct cases are investigated: the reflection–transmission problem, band-structure analysis of one-dimensional inhomogeneous photonic crystals, and inhomogeneous planar waveguide analysis.

A. Reflection–Transmission Analysis

In this subsection appropriate boundary conditions for the analysis of reflection and transmission from an inhomogeneous layer, when a lightwave is obliquely incident from region I, is discussed. In this case we have

$$U_1 = 1 + R,$$

$$V_1 = k_{x1}(1 - R), \quad (17)$$

$$U_3 = T,$$

$$V_3 = k_{x3} T, \quad (18)$$

where R and T are reflection and transmission coefficients, respectively, and k_{x1} and k_{x3} are normal wave vectors in region I and region III, respectively. The amplitude of the incident wave is normalized to unity.

Substituting Eqs. (17) and (18) in Eqs. (15) and (16), one can obtain appropriate boundary equations. Simultaneously solving the set of equations from Eqs. (11), (15), and (16), one can determine the values of R, T, and q_m. Note that the set of equations to be solved are algebraic equations, which can be easily handled.

B. Band-Structure Analysis

Inhomogeneous photonic crystals have been the subject of considerable attention in the literature.[6] An inhomogeneous one-dimensional photonic crystal is simply constructed by periodical repetition of an inhomogeneous layer such as the one depicted in Fig. 1. For analyzing such structures, Bloch boundary condition must be applied. The Bloch boundary condition calls for

$$\begin{bmatrix} U_3 \\ V_3 \end{bmatrix} = \exp(-j\kappa d) \begin{bmatrix} U_1 \\ V_1 \end{bmatrix}. \quad (19)$$

Two normalized quantities are introduced: normalized frequency defined by $\omega_n = (\omega/c)d$ and a normalized Bloch wave vector defined by $\kappa_n = \kappa d$. One obtains band structure by plotting ω_n versus κ_n. By using the relation in Eq. (19) between electromagnetic fields at $x=0$ and $x=d$ and Eqs. (11), (15), and (16), one can easily obtain the band structure by scanning ω_n and calculating the corresponding values of κ_n. This can be done in two ways, first, by setting the determinant of the coefficient matrix of the resulting set of unforced equations to zero and, second, by organizing the equations in an eigenvalue problem form. The latter is accomplished by one's obtaining the transfer matrix of the inhomogeneous layer in terms of Legendre polynomials. Equations (11), (15), and (16) are repeated here in their matrix form:

$$[A][\bar{q}_m] = 0, \quad (20)$$

$$[B][\bar{q}_m] = \begin{bmatrix} U_1 \\ V_1 \end{bmatrix}, \quad (21)$$

$$[C][\bar{q}_m] = \begin{bmatrix} U_3 \\ V_3 \end{bmatrix}. \quad (22)$$

$[A]$ is a $(M-1)\times(M+1)$ matrix, and $[B]$ and $[C]$ are $2\times(M+1)$.

The transfer matrix of the inhomogeneous layer is therefore obtained as

$$[Q] = [C_{2\times(M+1)}] \left[\frac{A_{(M-1)\times(M+1)}}{B_{2\times(M+1)}} \right]^{-1} \left[\frac{[0]_{(M-1)\times(M+1)}}{I_{2\times 2}} \right], \quad (23)$$

where $[A_{(M-1)\times(M+1)}/B_{2\times(M+1)}]$ represents a matrix whose upper part is filled with the matrix $[A]$ and its lower part is filled with $[B]$, and [0] is a null matrix of $(M-1)\times(M+1)$ dimensions.

The transfer matrix relates the electromagnetic fields at $x=0$ to the fields at $x=d$:

$$\begin{bmatrix} U_3 \\ V_3 \end{bmatrix} = [Q] \begin{bmatrix} U_1 \\ V_1 \end{bmatrix}. \quad (24)$$

By using the transfer matrix, it is easy to show that

$$\cos(\kappa_n) = \frac{q_{11} + q_{22}}{2}, \quad (25)$$

where q_{11} and q_{22} are the diagonal terms of $[Q]$. Note that the transfer matrix of the inhomogeneous layer is obtained holistically, without dividing the structure into homogeneous sublayers.

C. Waveguide Analysis

There has been a considerable amount of study on the confined modes of inhomogeneous slab waveguides. This is due to the advancements in fabrication techniques, especially the diffusion technique, and the need to control dispersion properties of optical waveguides. For analyzing bounded states of planar inhomogeneous waveguides, boundary conditions are

$$V_1 = j\gamma_1 U_1, \quad (26)$$

$$V_3 = -j\gamma_3 U_3, \quad (27)$$

where $\gamma_1 = \sqrt{\beta^2 - k_1^2 n_1^2}$ and $\gamma_3 = \sqrt{\beta^2 - k_1^2 n_3^2}$.

By using the relations of Eqs. (26) and (27) in Eqs. (15) and (16) and solving with Eq. (11), one can calculate the values of β for each frequency and, consequently, obtain the dispersion diagram of the inhomogeneous waveguide. In the subsequent examples on waveguide analysis, the normalized frequency is $\Omega_n = d/\lambda$, and the effective index is defined as $N_{\text{eff}} = \beta d/2\pi\Omega_n$.

3. RESULTS AND EXAMPLES

Examples and results are presented in three distinct parts: first, two examples on the calculation of reflection and transmission coefficients of inhomogeneous layers are given; second, some examples on the band-structure calculation of inhomogeneous one-dimensional photonic crystals are given; and, finally, examples on guided-mode analysis of inhomogeneous waveguides are presented. The results have been verified by using exact solutions, wherever available, and other well-known numerical methods. In each case, a comparison has been made between the proposed method and some other approaches.

As the first example, a linear profile, shown in Fig. 2, is investigated. The refractive index linearly grows from n_1 at $x=0$ to n_2 at $x=d$. The normalized reflectance of this

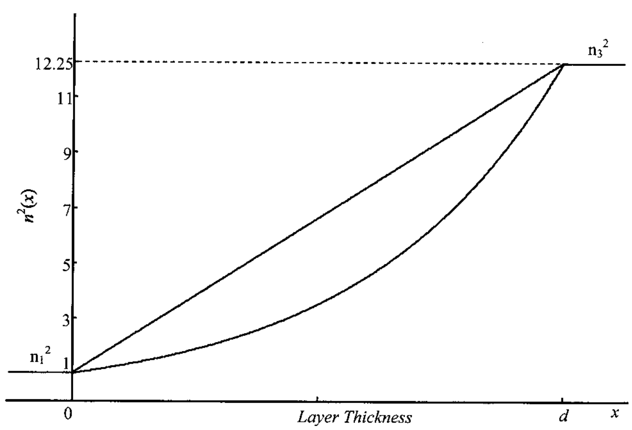

Fig. 2. Linear and exponential permittivity profiles.

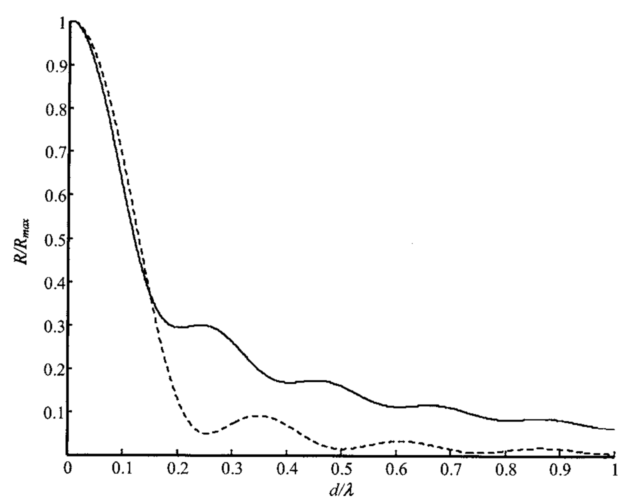

Fig. 3. Reflectance versus d/λ for the linear permittivity profile (solid curve) and exponential permittivity profile (dashed curve).

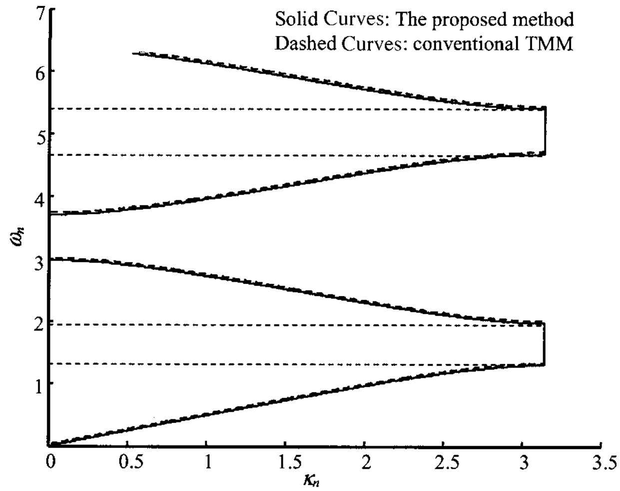

Fig. 4. Band structure of the photonic crystal with an exponential profile unit cell.

linear profile (solid curve) is plotted in Fig. 3 versus d/λ. It is obvious that only two terms in the expansion of Eq. (10) are sufficient to perfectly describe this linear profile. An exact analytic solution is available for reflectance of this structure.[19] With 13 polynomial terms ($M=13$), the results are obtained with the maximum absolute error of 2.53×10^{-4}. Another example is an inhomogeneous layer of exponential profile shown in Fig. 2. The refractive index profile varies as

$$n(x) = n_1 \exp\left[\frac{x}{d}\ln\left(\frac{n_3}{n_1}\right)\right]$$

from $x=0$ to n_2 at $x=d$. This profile can be approximated well by six terms in Eq. (10). The normalized reflectance of this exponential profile is shown in Fig. 3 (dashed curve). The results obtained by using 15 Legendre polynomial terms ($M=15$), agrees well with the exact solution; the maximum absolute error is 3.54×10^{-5}. Note that any desired level of accuracy can be obtained by one's keeping an appropriate number of polynomial terms.

The second part of the examples is devoted to band-structure analysis of inhomogeneous one-dimensional photonic crystals. The first example is a photonic crystal whose unit cell has the exponential profile shown in Fig. 2. The corresponding band structure for the angle of incidence equal to 42° is shown in Fig. 4, and the normalized group velocity is shown in Fig. 5. The results are obtained by using six terms of Eq. (10) and 15 Legendre polynomial terms, as before.

The conventional TMM is also applied for calculating this band structure. Using the TMM, we obtain converged results by dividing the unit cell into 400 homogeneous sublayers. In this example, only four bands are dealt with; however, the convergence of TMM results is much slower for higher-order bands. As the number of sublayers increases, the computation time is drastically increased, whereas the proposed method yields converged results efficiently. A comparison of computation times of these two approaches is conducted in Table 1.

Another example is the band-structure calculation of a photonic crystal with a chirped unit cell, shown in Fig. 6.

The angle of incidence is 30°. Note that the profile shown in Fig. 6 has a nearly harsh variation and is discontinuous between any two succeeding cells because the value of the refractive index at $x=0$ does not coincide with that at $x=d$. Obviously, interpolating such a profile with expansion in Eq. (10) requires keeping many terms. It can be easily checked that 40 terms in Eq. (10) yield an acceptable approximation of the profile. Band structure and group-velocity variation of this photonic crystal are shown in Figs. 7 and 8, respectively. The results of the polynomial expansion (solid curves), have been obtained by keeping 27 polynomial terms ($M=27$). And the results of the TMM are obtained by dividing the unit cell into 300 sublayers. A comparison of the computation times are given in Table 1. To further verify the results, we employ the effective index method[34] in the well-known Bragg formula to obtain the center of the bandgaps as

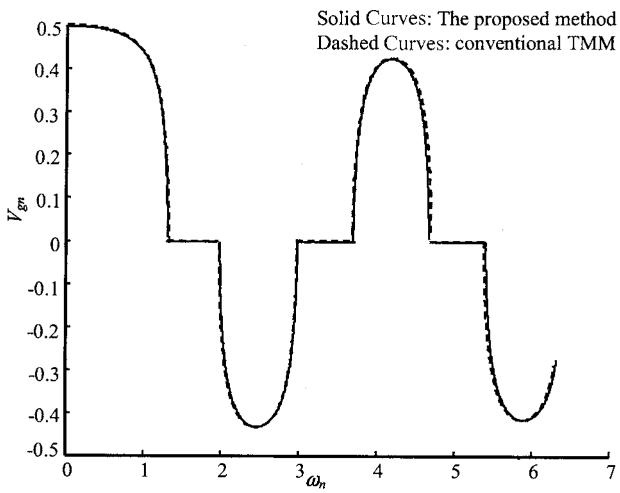

Fig. 5. Normalized group velocity versus the normalized frequency for the photonic crystal with an exponential profile unit cell.

Table 1. Comparison of Computation Times for Band-Structure Calculation Using the Proposed Method and TMM

Method	Exponential Profile (s)	Chirped Profile (s)
Conventional TMM	114.25	51.06
Proposed method	7.06	30.5

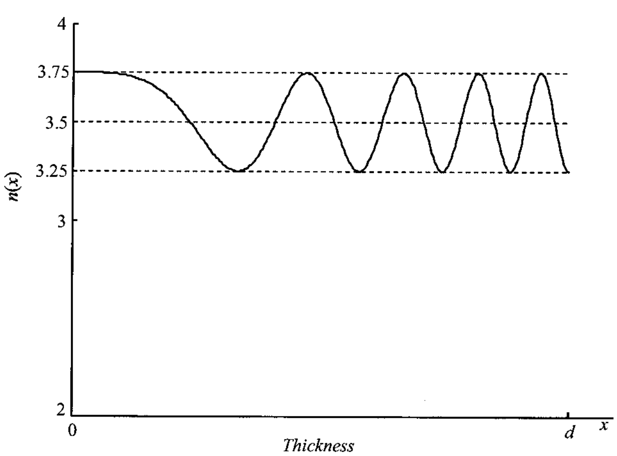

Fig. 6. Chirped refractive index profile.

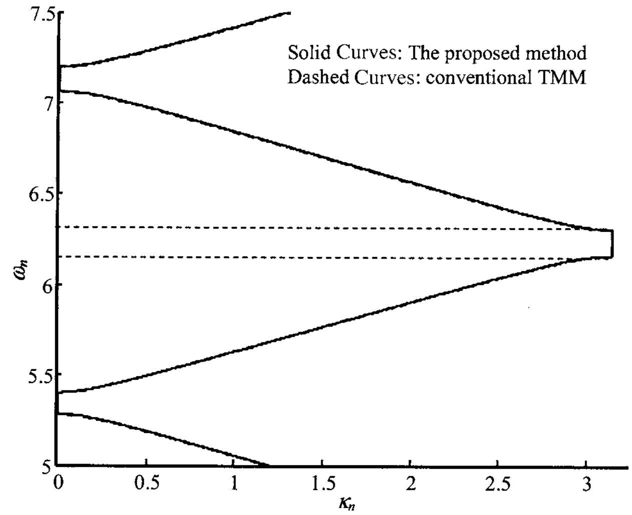

Fig. 7. Band structure of a chirped photonic crystal.

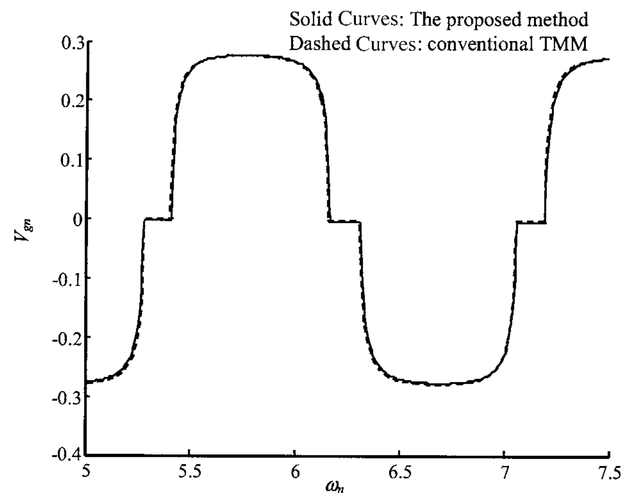

Fig. 8. Normalized group velocity versus the normalized frequency for the chirped photonic crystal.

$$\omega_n = \frac{m\pi}{n_{\text{eff}}\cos(\theta_{\text{eff}})}, \tag{28}$$

where m is the number of the bandgap. The results are summarized in Table 2, which agree well with those depicted in Figs. 4 and 7.

The final examples are given on a guided-mode calculation of an inhomogeneous slab waveguide. A parabolic profile waveguide shown in Fig. 9 is analyzed. This profile can be exactly described by three terms of expansion in Eq. (10). The dispersion diagram of this waveguide, shown in Fig. 10, is obtained by calculating the corresponding values of β_n in each normalized frequency, The results agree well with those reported in Ref. 17 at $\Omega_n=2.3857$. The dispersion diagram calculated by keeping eight polynomial terms ($M=8$) agrees within a maximum difference of 1% with that obtained by the TMM using 200 sublayers. Also, for the sake of further verification, guided modes calculated at $\Omega_n=2.3857$ by using the polynomial expansion method, the TMM, and the WKB method are compared in Table 3. Note that calculating the overall dispersion diagram for this waveguide by using the WKB method is cumbersome, since for the guided modes whose effective indices are near or at the discontinuity the conventional phase shift of $\pi/2$ at turning points does not work, and appropriate phase shifts must be substituted.[15]

Table 2. Forbidden Frequencies Calculated Using the Bragg Formula Corresponding to the Band Structures Shown in Figs. 4 and 7

Method	n_{eff}	θ_{eff} (rad)	ω_n
Exponential profile	1.9956	0.3419	1.671 ($m=1$)
Chirped profile	3.5295	0.0354	5.3439 ($m=6$)

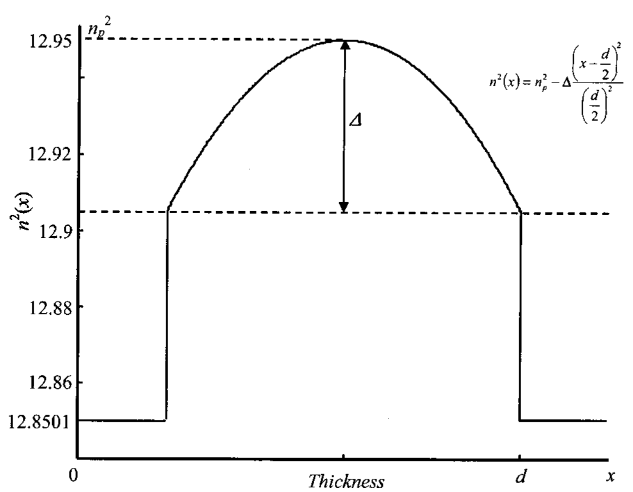

Fig. 9. Parabolic permittivity profile waveguide with $n_p=3.5986$ and $\Delta=0.0448$.

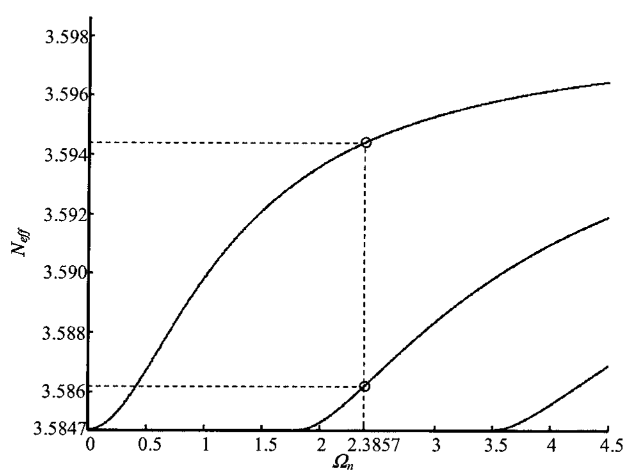

Fig. 10. Parabolic waveguide dispersion diagram.

Table 3. Comparison of the Effective Indices of the Parabolic Waveguide Shown in Fig. 9 at $\Omega_n=2.3857$, Obtained by Using the Proposed Method, Conventional TMM, and WKB Method

Method	Effective Index (First Mode)	Effective Index (Second Mode)
Proposed method	3.586176	3.594406
Conventional TMM (200 sublayers)	3.586170	3.594396
WKB	—	3.594673

Table 4. Comparison of the Computation Times of the Polynomial Expansion, TMM, and WKB, Corresponding to the Parabolic Waveguide Shown in Fig. 9

Method	Overall Dispersion Diagram Computation Time (Frequency Steps=0.01) (s)	Second Mode Computation Time at a Single Frequency $\Omega_n=2.3857$ (s)
Proposed method	1399	0.85
Conventional TMM (200 sublayers)	5948	3.28
WKB	—	4

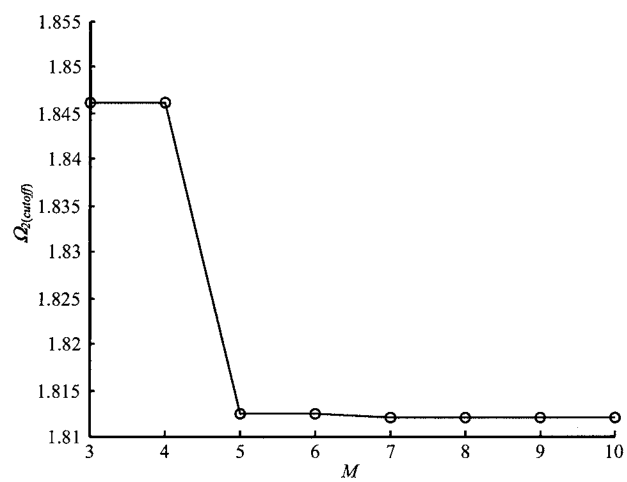

Fig. 11. Convergence of the second mode cutoff frequency versus the number polynomial basis functions.

Also, since this waveguide has a discontinuous refractive index profile, the WKB results lose their accuracy near the cutoff frequencies.[15] The computation times corresponding to the polynomial expansion method, the TMM, and the WKB method are compared in Table 4.

For showing the convergence rate of the proposed polynomial-based approach, the cutoff frequency corresponding to the second guided mode is plotted versus the polynomial basis functions in Fig. 11; it can be seen that after six polynomial terms ($M\geqslant 6$) are used the cutoff frequency converges within 2% of its final value. With the proposed method, the effect of different parameters of the waveguide refractive index profile on the frequency range over which the waveguide is single mode can be easily investigated. Here an analytic expression is given that describes the dependency of the second mode cutoff frequency on Δ, the parabolic profile height, shown in Fig. 9:

$$\Omega_{2(\mathrm{cutoff})} = 9.549\left[\frac{0.154 - 0.820\Delta - (0.526\Delta^2 - 0.174\Delta + 0.0161)^{1/2}}{19\Delta^2 - 10.2\Delta + 0.989}\right]^{1/2} \quad 0 < \Delta \leqslant 0.127. \tag{29}$$

By appropriately choosing Δ in Eq. (29), one can adjust the frequency range over which the waveguide is single mode. Equation (29) is obtained by six polynomial terms, and it can be easily shown that $\Omega_{c(\max)}$ is 2.6281 and occurs at $\Delta = 0.1262$.

4. CONCLUSIONS

In this paper a new method for analyzing inhomogeneous optical structures based on the Legendre polynomial expansion of electromagnetic fields is reported. Different problems including calculation of reflection–transmission coefficients of inhomogeneous layers, band-structure calculation of inhomogeneous one-dimensional photonic crystals, and eigenmode extraction of inhomogeneous waveguides are investigated. In this approach, the electromagnetic fields expanded in terms of orthogonal Legendre polynomials are substituted in the Helmholtz equation, where the $(\mathrm{d}^2/\mathrm{d}x^2)U(x)$ and $k^2n^2(x)$ terms are analytically expanded in terms of Legendre polynomials by using the relevant properties, i.e., Eqs. (12) and (6), and interpolating the permittivity profile. These analytic expressions yielding the matrix form of Eq. (11) considerably augment the efficiency of the proposed method. This approach yields algebraic, easy-to-handle equations. Especially, in the case of eigenmode analysis of inhomogeneous waveguides, instead of transcendental dispersion equations, algebraic dispersion equations are derived, which can be easily solved. The presented approach can be used to obtain the transfer matrix of an arbitrary inhomogeneous layer holistically, eliminating the need for dividing the refractive index or permittivity profile into homogeneous sublayers. Comparisons with other well-known methods are made, and the results are verified. This approach benefiting from the regular behavior of polynomials is versatile and can be systematically applied to any arbitrary inhomogeneous structure. For many practical structures, this approach works quickly and efficiently, and this is particularly suitable for fast calculation of reflection–transmission coefficients for real-time monitoring of fabrication processes. Also, with the proposed method, approximate analytic expressions describing the effects of different parameters on the optical response of the structure can be derived; such analytic expressions are valuable for synthesis and optimization purposes.

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