Custom Reduction of Arithmetic in Linear DSP Transforms

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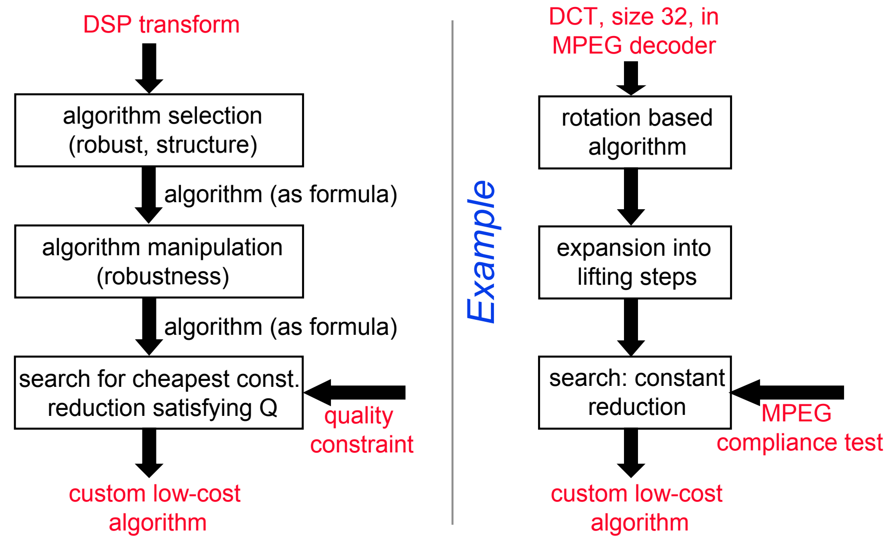
Research Overview

- Linear DSP transforms
  - e.g. DFT, DCTs, WHT, DWTs, ….
  - ubiquitously used, often in computation intensive kernels
  - comprised of additions and multiplication-by-constant
  - applications: multimedia, bio-metric, image/data processing . . . .
- Light-weight hardware implementations
  - fixed-point data format
  - multiplierless: mult-by-constant as shifts and adds
  - problem 1: output quality reduced by cost-saving measures
    (reducing the bitwidth of data and constants)
  - problem 2: different applications have vastly different quality metric and requirements
    ⇒ need application specific tuning

Our Goal: automatic, custom reduction of arithmetic (additions) w.r.t. a given application's requirements

Our Automatic Flow



Related Work

- Liang/Tran, “Fast Multiplierless Approximation of the DCT with the Lifting Scheme,” IEEE Trans. Sig. Proc., 49(12) 2001, pp. 3032-3044
  - examined arithmetic cost reduction for DCT size 8
  - steps performed by hand, exhaustive search
- Fang/Rutenbar/Püschel/Chen, “Toward Efficient Static Analysis of Finite-Precision Effects in DSP Applications via Affine Arithmetic Modeling,” Proc. DAC 2003
  - efficient static analysis of output error (hard and probabilistic)
  - range of input values used/needed
  - analysis assumes a common global bitwidth
- Püschel/Singer/Voronenko/Xiong/Moura/Johnson/Veloso/Johnson, “SPIRAL system”, www.spiral.net
  - automatic generation of custom runtime optimized DSP transform software
  - provides implementation environment for our approach (in particular algorithm generation and manipulation)

Outline

- DSP transform algorithms
- Algorithm manipulation for robustness
- Multiplication by constants
- Search Methods
- Results

DSP Algorithms as Formulas: Example DFT size 4

Cooley/Tukey FFT (size 4):

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Fourier transform **Diagonal matrix (twiddles)**

$$DFT_4 = (DFT_2 \otimes I_2) \cdot diag(1,1,1,i) \cdot (I_2 \otimes DFT_2) \cdot [(2,3),4]$$

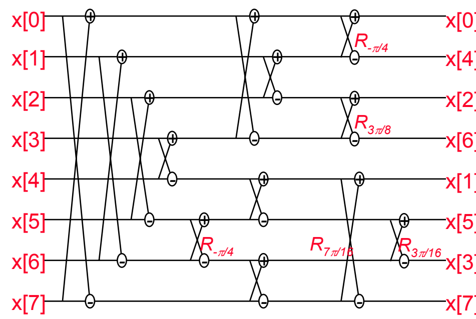
Kronecker product **Identity** **Permutation**

allows for computer generation/manipulation (provided by SPIRAL)

Example: DCT size 8

$$
\begin{aligned}
&[(2,5)(4,7)(6,8),8] \\
&\cdot(diag(1,1/\sqrt{2}) \oplus R_{3\pi/8} \oplus R_{15\pi/16} \oplus R_{21\pi/16}) \\
&\cdot[(2,4,7,3,8),8] \cdot ((DFT_2 \otimes I_3) \oplus I_2) \cdot [(5,6),8] \\
&\cdot(I_4 \oplus 1/\sqrt{2} \cdot DFT_2 \oplus I_2) \cdot [(2,3,4,5,8,6,7),8] \\
&\cdot(I_2 \otimes ((DFT_2 \oplus I_2) \cdot [(2,3),4] \cdot (I_2 \otimes DFT_2))) \\
&\cdot[(1,8,6,2)(3,4,5,7),8]
\end{aligned}
$$

as formula
(generated by SPIRAL)



as data flow diagram

Basic building blocks:

- 2 x 2 rotations, DFT_2's (butterflies), permutations, diagonal matrices (scaling)

Algorithm is orthogonal = robust to input errors (from fixed point representation)

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Fixed Point Error: Data vs. Transform

Implementing a transform $x \mapsto Tx$ in fixed point arithmetic produces two type of errors:

- Error in input x: $\| x - \tilde{x} \|$
  - from rounding of the input coefficients *x* to the fix-point data representation $\tilde{x}$
  - for robustness: choose orthogonal algorithms
- Error in transform: $\| T - \tilde{T} \|$
  - from finite precision multiplication by constants

    further approximation is a source of savings in multiplierless implementations
  - for robustness: translate algorithm into lifting steps

Lifting Steps

- Lifting step (LS): $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$
  - invertible (det = 1) independent of approximation of x, y
  - inverse of LS is also LS *(with –x, -y)*

    ∴ *if LS is cheap, then so is its inverse*
- Rotation as lifting steps

target for approximation

$$R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$$

$$p = \frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}, \quad u = -\sin\alpha$$

➡ rotation based algorithms can be automatically expanded into LS

Error Analysis

- rounding error in the first lifting step (third LS analogous)

$$\widetilde{R}_\alpha = R_\alpha + \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = R_\alpha + \begin{bmatrix} -\varepsilon \sin\alpha & \varepsilon \cos\alpha \\ 0 & 0 \end{bmatrix}$$

not magnified

- rounding error in the second lifting step

$$\widetilde{R}_\alpha = R_\alpha + \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = R_\alpha + \begin{bmatrix} \varepsilon \tan\frac{\alpha}{2} & \varepsilon \tan^2\frac{\alpha}{2} \\ \varepsilon & \varepsilon \tan\frac{\alpha}{2} \end{bmatrix}$$

ε is magnified, unless α in $[0, \pi/2]$ or $[3\pi/2, 2\pi]$

Solution: angle manipulation

$$R_\alpha = R_{\alpha-\pi/2} \cdot R_{\pi/2} = R_{\alpha-\pi/2} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Ensuring Robustness

Steps to ensure robustness

- Choose algorithms based on rotations
- Manipulate angles of rotations
- Expand into lifting steps

➡ *Done automatically as formula manipulation*

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Multiplication by Constants

Operations in transforms:

$y = x_1 + x_2$ additions

$y = cx$ multiplication by constant

Example:

simple $c=0.10111011 =$ ➡ 5 adds (5 shifts)

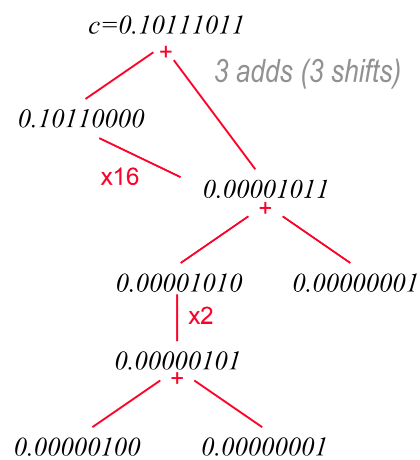
SD recoding 1 $c=0.1100\bar{1}10\bar{1}$ ➡ 4 adds (3 shifts)

SD recoding 2 $c=0.11000\bar{1}0\bar{1}$ ➡ 3 adds (3 shifts)

SD recoding is not optimal

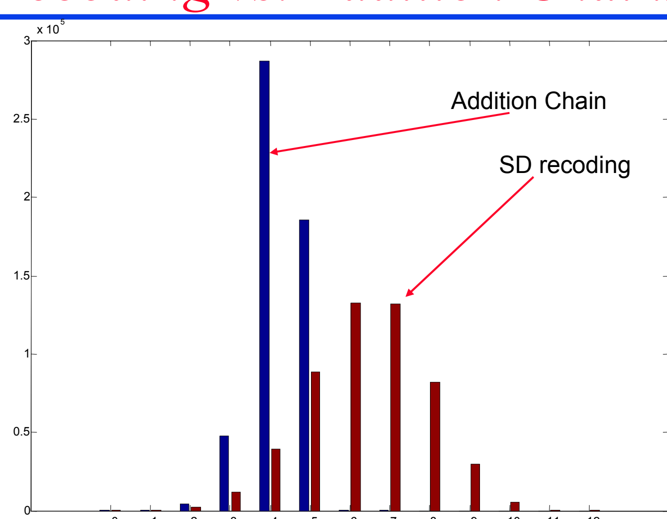
Addition/Subtraction Chain

- Provide optimal solution for constant mult using adds and shifts
- Finding the optimal addition chain is a hard problem
- A near optimal table of solutions can be computed using dynamic programming methods*
- For all constants up to 2^{19}
  - only 225 constants require more than 5 additions
    (214@6, 11@7)



**Sebastian Egner, Philips Research, Eindhoven*

SD recoding vs. Addition Chains



Histogram of addition cost for all constants between 1 and 2^{19}

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Optimization Problem

Given a linear DSP transform and quality measure *Q*

1. Find the multiplierless implementation with the least arithmetic cost *C* (number of additions) that satisfies a given *Q* threshold

2. Find the multiplierless implementation with the highest quality *Q* for a given arithmetic cost *C* threshold

Quality Measures of Transforms

For an approximation $\widetilde{T}$ of a transform T.

- Transform independent Q
  - $\| T - \widetilde{T} \|$ for some norm $\| \cdot \|$
- Transform dependent Q
  - coding gain for DCT
  - convolution error for DFT
- Application-based Q
  - MPEG standard compliance test

Search Space: approximating multiplicative constants

- For each multiplication-by-constant in the transform choose custom bitwidth $i \in [0 \ldots k-1]$
  - Given n constants, k^n configurations are possible
- But, for a given constant, not all k configurations lead to different cost,

e.g., given 5-bit constant 0.11101, SD recoding gives

5-bit = .11101	= $1.00\bar{1}01$	$\Rightarrow$ 2 adds
4-bit = .1110	= $1.00\bar{1}0$	$\Rightarrow$ 1 adds
~~3-bit = .111~~	~~= $1.00\bar{1}$~~	~~$\Rightarrow$ 1 adds~~
~~2-bit = .11~~	~~= 0.11~~	~~$\Rightarrow$ 1 adds~~
1-bit = .1	= 0.1	$\Rightarrow$ 0 adds
0-bit = 0	= 0	$\Rightarrow$ 0 adds

Recall all constants up to 19-bits can be reduced to 5 adds

Search Methods

- Global Bitwidth
  - all constant assigned the same bitwidth
  - *very fast (small search space), but only works well in some cases*
- Greedy Search
  - starting with maximum bitwidth, in each round, choose one constant to be reduced by 1-bit that minimizes quality loss
    (also go bottom-up instead of top-down)
  - *local minima traps are possible*
- Evolutionary Search
  - start with a population of random configurations
  - in each round
    1. breed a new generation by crossbreeding and mutations
    2. select from generation the fittest members
    3. repeat new round
  - *local minima traps*

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Interaction between Transforms, Q and Search

- Goal: given a transform and a required Q threshold, find an approximation to the transform that requires the fewest additions
- Transforms and Q tested

Transform	*Quality Threshold*
8-pt. DCT-II	8.82 dB coding gain (cg)
16-pt. DFT	Convolution error = 1
32-pt. DCT-II	Limited Compliance (LC) MP3 decoder♣
18x36 IMDCT	LC MP3 decoder♣

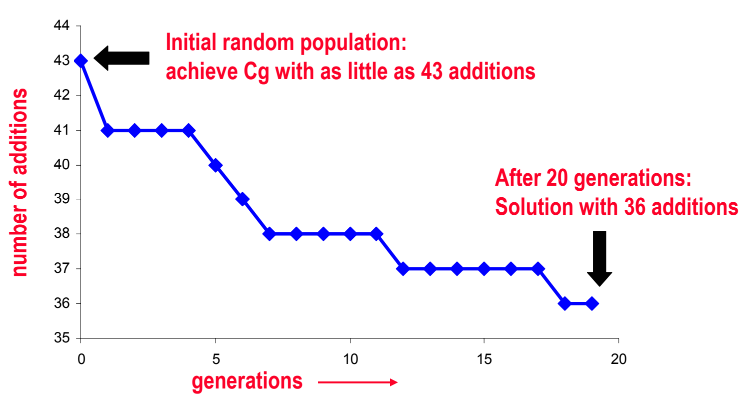
- 3 searches methods were compared
- entire framework implemented as part of SPIRAL (www.spiral.net)

♣*MAD Decoder by Robert Mars, http://www.underbit.com/products/mad*

Example: Evolutionary Search

Evolutionary Search DCT of size 8 with 12 constants

- Q = cg > 8.82, exact DCT has 8.8259
- constant bit length in [0..31]



Choosing 31 bits for all constants: 126 additions

Summary of Search Comparison

	Number of Additions (fewer is better)			
	8 pt. DCT-II (8.82 dB cg)	16 pt. DFT (conv. err = 1)	32 pt. DCT-II (LC MP3)	18x36 IMDCT (LC MP3)
initial (31 bits)	126	500	1222	643
global	40	168	*408*	182
evol.	*36*	185	490	212
greedy (top-down)	56	158	417	*170*
greedy (bottom-up)	57	*154*	n/a	n/a

One search method alone is not sufficient — each search performs differently depending on transform and quality measure

Approximation of DCT within JPEG

- Approximate DCT-II inside JPEG while retain images of reasonable quality
  - Q = Peak Signal to Noise Ratio (decibels) of decompressed JPEG image against the original uncompressed input image.

$$\mathrm{PSNR} = 20 \times \log_{10}\left(\frac{255}{\mathrm{RMSE}}\right)$$

$$\mathrm{RMSE} = \sqrt{\frac{1}{512 \times 512}\sum_{i}^{512}\sum_{j}^{512}\left[D(i,j) - O(i,j)\right]^2}$$

  - Q Threshold
    - Test Image: Lena, 512x512 pixel, 8-bit grayscale
    - PSNR must be at least 30 decibels or image becomes noticeably lossy).

Approximation of DCT within JPEG

- Before approximating, the original DCT♣ requires 261 additions and produces a Lena image with a PSNR of 37.6462 dB.

Method	# Additions	PSNR
global	37	30.0354
evolutionary	67	36.5323
greedy (t-d)	*28*	*32.4503*

- Compare constants global vs. greedy search:
  - Global: [3/2, 3/2, 3/2, 3/2, 3/2, 3/2, 3/2, 1/2, -1/2, 1,
    -1/2, -1/2, 1/2, -1/2, -1, 1, -1, -1/4, 1/2, -1/4]
  - Greedy: [3/2, 1, 1, 1, 1, 1, 1, 1/2, -1/2, 1, -1/2,
    0, 1/2, 0, -1, 1, -1, 0, 1/2, -1/4]
  - Greedy succeeds in zeroing 3 constants that affect the high frequency (HF) outputs 'thrown away' by JPEG

♣*Base on source from Independent JPEG Group (IJG), http://www.ijg.org*

Summary

- Application specific tuning yields ample opportunities for optimization
- The optimization flow can be automated
  - algorithm selection and manipulation
  - arithmetic reduction through search
  - arbitrary quality measures supported
- Details of the arithmetic reduction is non-trivial
  - non-monotonic relation between Q and C
  - different search methods succeed in different scenarios
- The results of this study needs to be combined with other aspects of DSP domain-specific high-level synthesis