# SPIRAL
# Automating High Quality Software Production

## Tutorial at HPEC 2019

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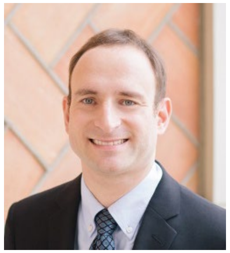

Franz Franchetti

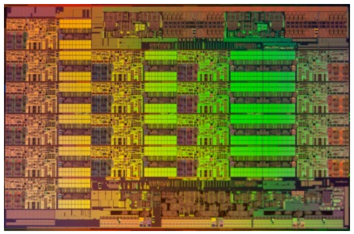

Tze Meng Low


Mike Franusich

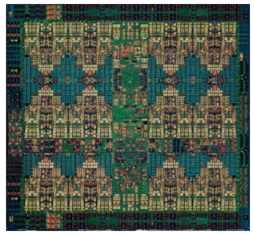
Tutorial based on joint work with the Spiral team at CMU, UIUC, and Drexel

# Today’s Computing Landscape

1 Gflop/s = one billion floating-point operations (additions or multiplications) per second

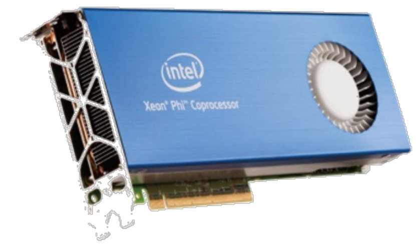


**Intel Xeon 8180M**
***2.25 Tflop/s, 205 W***
28 cores, 2.5—3.8 GHz
2-way—16-way AVX-512



**IBM POWER9**
***768 Gflop/s, 300 W***
24 cores, 4 GHz
4-way VSX-3

**Nvidia Tesla V100**
***7.8 Tflop/s, 300 W***
5120 cores, 1.2 GHz
32-way SIMT



**Intel Xeon Phi 7290F**
***1.7 Tflop/s, 260 W***
72 cores, 1.5 GHz
8-way/16-way LRBni

**Snapdragon 835**
***15 Gflop/s, 2 W***
8 cores, 2.3 GHz
A540 GPU, 682 DSP, NEON

**Intel Atom C3858**
***32 Gflop/s, 25 W***
16 cores, 2.0 GHz
2-way/4-way SSSE3

**Dell PowerEdge R940**
***3.2 Tflop/s, 6 TB, 850 W***
4x 24 cores, 2.1 GHz
4-way/8-way AVX

**Summit**
***187.7 Pflop/s, 13 MW***
9,216 x 22 cores POWER9
+ 27,648 V100 GPUs

# 2019: What $1M Can Buy You



**Dell PowerEdge R940**
***4.5 Tflop/s, 6 TB, 850 W***
4x 28 cores, 2.5 GHz


**OSS FSAn-4**
***200 TB PCIe NVMe flash***
80 GB/s throughput

**BittWare TeraBox**
***18M logic elements, 4.9 Tb/sec I/O***
8 FPGA cards/16 FPGAs, 2 TB DDR4

**AberSAN ZXP4**
***90x 12TB HDD, 1 kW***
1PB raw


**Nvidia DGX-1**
***8x Tesla V100, 3.2 kW***
170 Tflop/s, 128 GB

# ISAs Longevity and Abstraction Power

## F-16A/B, C/D, E/F, IN, IQ, N, V: Flying since 1974


## Compare: Desktop/workstation class CPUs/machines

**Assembly code compatible !!** 7

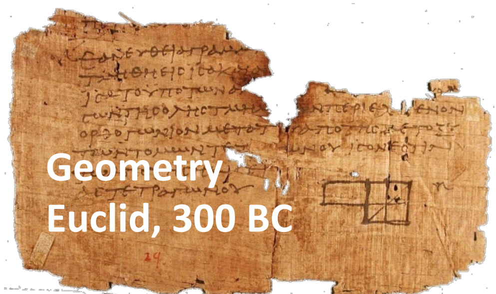


**x86 binary compatible, but 500x parallelism ?!**

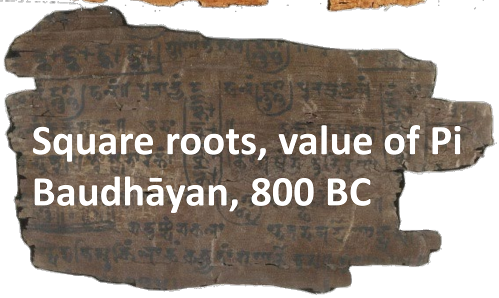
| 1972 | 1989 | 1994 | 2006 | 2011 | 2018 |
|---|---|---|---|---|---|
| Intel 8008 | IBM PC/XT compatible | IBM RS/6000-390 | GeForce 8800 | Xeon Phi | Xeon Platinum 8180M |
| 0.2—0.8 MHz | 8088 @ 8 MHz, 640kB RAM | 256 MB RAM, 6GB HDD | 1.3 GHz, 128 shaders | 1.3 GHz, 60 cores | 28 cores, 2.5-3.6 GHz |
| Intelligent terminal | 360 kB FDD, 720x348 mono | 67 MHz Power2+, AIX | 16-way SIMT | 8/16-way SIMD | 2/4/8/16-way SIMD |

**$10^7$ – $10^8$ compounded performance gain over 45 years**

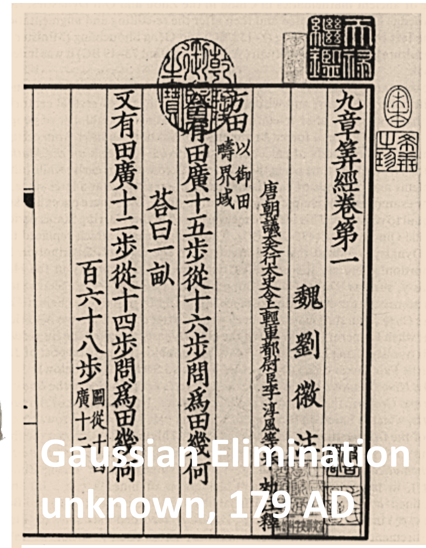
# Algorithms and Mathematics: 2,500+ Years



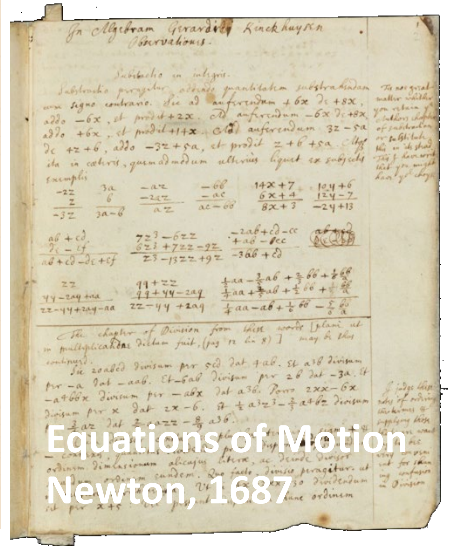
Geometry
Euclid, 300 BC



Square roots, value of Pi
Baudhāyan, 800 BC



Gaussian Elimination
unknown, 179 AD

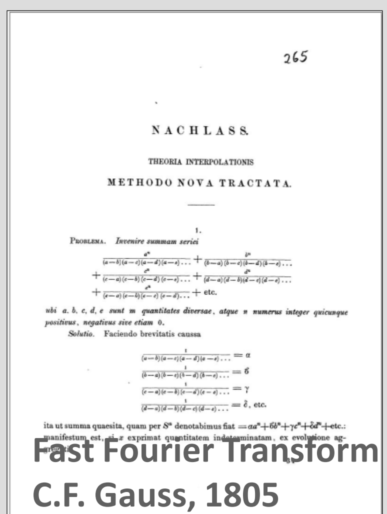


Equations of Motion
Newton, 1687

Bernoulli Numbers
Kōwa, 1712

Algebra
al-Khwārizmī, 830

## Fast Fourier Transform



265

NACHLASS.

THEORIA INTERPOLATIONIS

METHODO NOVA TRACTATA.

1.

Problema. *Invenire summam seriei*

$$\frac{a^n}{(a-b)(a-c)(a-d)(a-e)\dots}+\frac{b^n}{(b-a)(b-c)(b-d)(b-e)\dots}+\frac{c^n}{(c-a)(c-b)(c-d)(c-e)\dots}+\frac{d^n}{(d-a)(d-b)(d-c)(d-e)\dots}+\frac{e^n}{(e-a)(e-b)(e-c)(e-d)\dots}+\text{etc.}$$

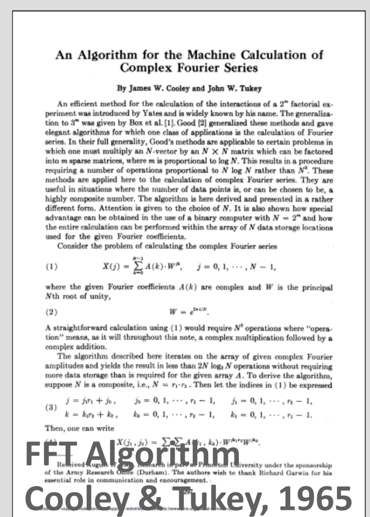
*ubi a, b, c, d, e sunt m quantitates diversae, atque n numerus integer quicunque positivus, negativus sive etiam 0.*

*Solutio.* Faciendo brevitatis caussa

$$\frac{1}{(a-b)(a-c)(a-d)(a-e)\dots}=\alpha$$
$$\frac{1}{(b-a)(b-c)(b-d)(b-e)\dots}=\beta$$
$$\frac{1}{(c-a)(c-b)(c-d)(c-e)\dots}=\gamma$$
$$\frac{1}{(d-a)(d-b)(d-c)(d-e)\dots}=\delta, \text{ etc.}$$

ita ut summa quaesita, quam per $S^n$ denotabimus fiat $=\alpha a^n+\beta b^n+\gamma c^n+\delta d^n+$etc.: manifestum est, si $x$ exprimat quantitatem indeterminatam, ex evolutione ag-

Fast Fourier Transform
C.F. Gauss, 1805



An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a $2^m$ factorial experiment was introduced by Yates and is widely known by his name. The generalization to $3^m$ was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an $N$-vector by an $N \times N$ matrix which can be factored into $m$ sparse matrices, where $m$ is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than $N^2$. These methods are applied here to the calculation of complex Fourier series. They are useful in situations where the number of data points is, or can be chosen to be, a highly composite number. The algorithm is here derived and presented in a rather different form. Attention is given to the choice of $N$. It is also shown how special advantage can be obtained in the use of a binary computer with $N = 2^m$ and how the entire calculation can be performed within the array of $N$ data storage locations used for the given Fourier coefficients.

Consider the problem of calculating the complex Fourier series

$$(1)\qquad X(j)=\sum_{k=0}^{N-1}A(k)\cdot W^{jk},\qquad j=0,1,\cdots,N-1,$$

where the given Fourier coefficients $A(k)$ are complex and $W$ is the principal $N$th root of unity,

$$(2)\qquad W=e^{2\pi i/N}.$$

A straightforward calculation using (1) would require $N^2$ operations where "operation" means, as it will throughout this note, a complex multiplication followed by a complex addition.

The algorithm described here iterates on the array of given complex Fourier amplitudes and yields the result in less than $2N \log_2 N$ operations without requiring more data storage than is required for the given array $A$. To derive the algorithm, suppose $N$ is a composite, i.e., $N = r_1 \cdot r_2$. Then let the indices in (1) be expressed

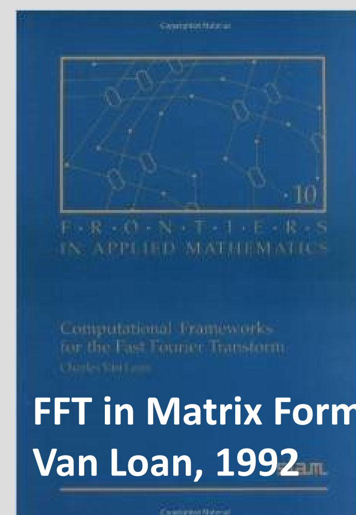
$$(3)\qquad \begin{aligned} j&=j_1r_1+j_0, & j_0&=0,1,\cdots,r_1-1, & j_1&=0,1,\cdots,r_2-1,\\ k&=k_1r_2+k_0, & k_0&=0,1,\cdots,r_2-1, & k_1&=0,1,\cdots,r_1-1.\end{aligned}$$

Then, one can write

$$X(j_1, j_0)=\sum\sum A(k_1, k_0)\cdot W^{jk_1r_2}W^{jk_0}.$$

Received August ..., Research in part at Princeton University under the sponsorship of the Army Research Office (Durham). The authors wish to thank Richard Garwin for his essential role in communication and encouragement.

FFT Algorithm
Cooley & Tukey, 1965




FFT in Matrix Form
Van Loan, 1992

# Programming/Languages Libraries Timeline

## Popular performance programming languages

- 1953: Fortran
- **1973: C**
- 1985: C++
- 1997: OpenMP
- 2007: CUDA
- 2009: OpenCL

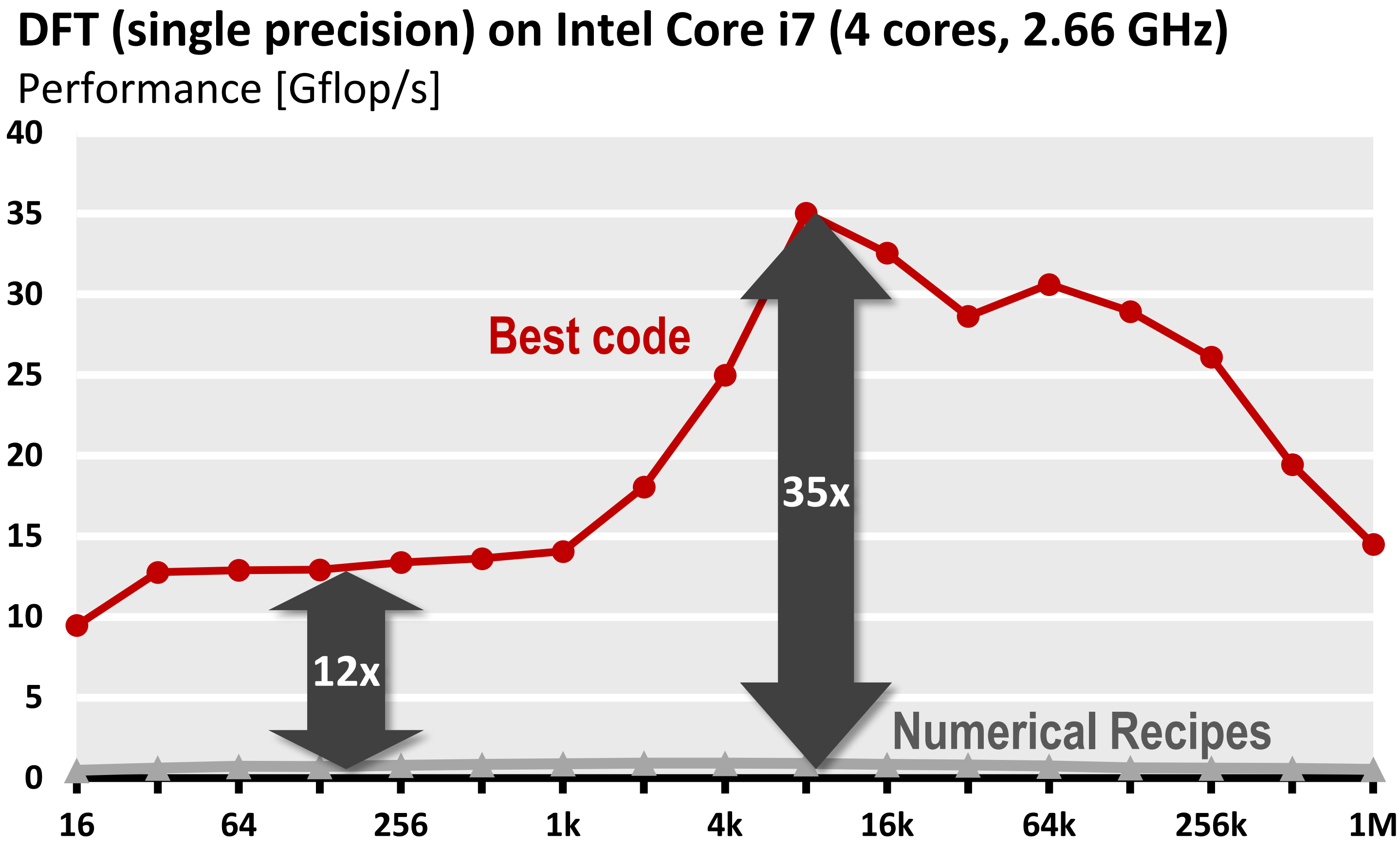
## Popular performance libraries

- 1979: BLAS
- 1992: LAPACK
- 1994: MPI
- 1995: ScaLAPACK
- 1995: PETSc
- 1997: FFTW

## Popular productivity/scripting languages

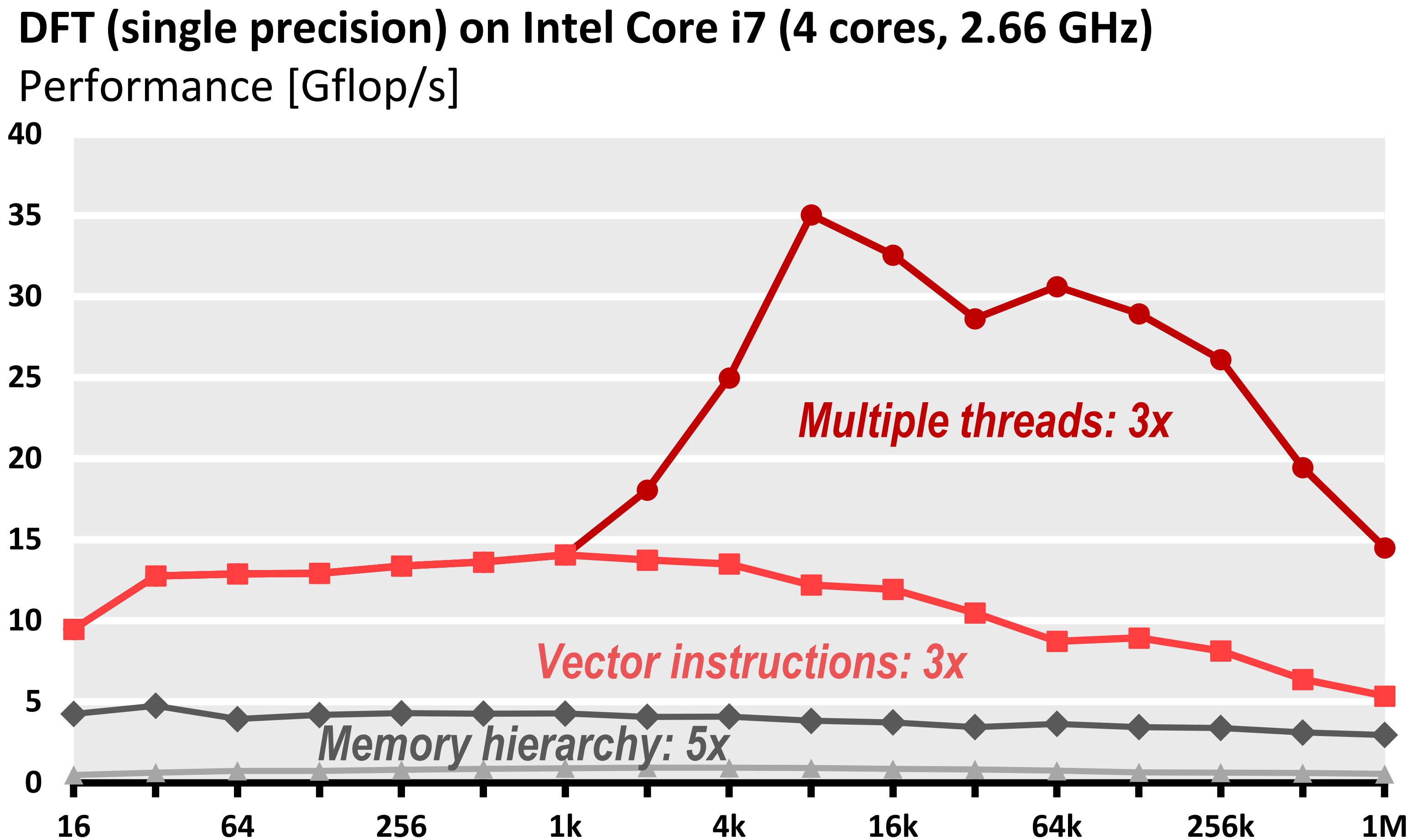
- 1987: Perl
- 1989: Python
- 1993: Ruby
- 1995: Java
- 2000: C#

# The Problem: Example DFT




- Standard desktop computer, cutting edge compiler, using optimization flags
- Implementations have same operations count: ≈$4n\log_2(n)$
- ***Same plots can be shown for all mathematical functions***

# DFT Plot: Analysis




***High performance library development has become a nightmare***

# How The Generated Code Looks Like

```
void dft64(float  *Y, float  *X) {
    __m512 U912, U913, U914, U915, U916, U917, U918, U919, U920, U921, U922, U923, U924, U925,...;
    a2153 = ((__m512  *) X);  s1107 = *(a2153);
    s1108 = *((a2153 + 4));   t1323 = _mm512_add_ps(s1107,s1108);
    ...
    U926 = _mm512_swizupconv_r32(_mm512_set_1to16_ps(0.70710678118654757),_MM_SWIZ_REG_CDAB);
    s1121 = _mm512_madd231_ps(_mm512_mul_ps(_mm512_mask_or_pi(
        _mm512_set_1to16_ps(0.70710678118654757),0xAAAA,a2154,U926),t1341),
        _mm512_mask_sub_ps(_mm512_set_1to16_ps(0.70710678118654757),0x5555,a2154,U926),
        _mm512_swizupconv_r32(t1341,_MM_SWIZ_REG_CDAB));
    U927 = _mm512_swizupconv_r32(_mm512_set_16to16_ps(0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757)),_MM_SWIZ_REG_CDAB);
    ...
    s1166 = _mm512_madd231_ps(_mm512_mul_ps(_mm512_mask_or_pi(_mm512_set_16to16_ps(
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757),
        0.70710678118654757, (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757)),
        0xAAAA,a2154,U951),t1362),
        _mm512_mask_sub_ps(_mm512_set_16to16_ps(0.70710678118654757,
        (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757), 0.70710678118654757,
        (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757), 0.70710678118654757,
        (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757), 0.70710678118654757,
        (-0.70710678118654757), 0.70710678118654757, (-0.70710678118654757)),0x5555,a2154,U951),
        _mm512_swizupconv_r32(t1362,_MM_SWIZ_REG_CDAB));
    ...
}
```

D. McFarlin, F. Franchetti, and M. Püschel: **Automatic Generation of Vectorized Fast Fourier Transform Libraries for the Larrabee and AVX Instruction Set Extension.** Proceedings of the 2009 High Performance Embedded Computing (HPEC), MIT Lincoln Laboratory. **Best paper award.**

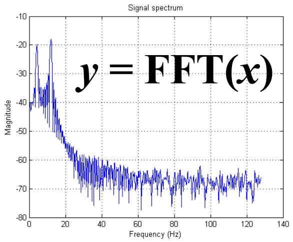
# Goal: Go from Mathematics to Software

## Given:

- **Mathematical problem specification**
  *does not change*
- **Computer platform**
  *changes often*

## Wanted:

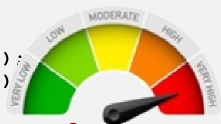
- **Very good implementation of specification on platform**
- **Proof of correctness**




*on*


**automatic** →

```
void fft64(double  *Y, double  *X) {
    ...
    s5674 = _mm256_permute2f128_pd(s5672, s5673, (0) | ((2) << 4));
    s5675 = _mm256_permute2f128_pd(s5672, s5673, (1) | ((3) << 4));
    s5676 = _mm256_unpacklo_pd(s5674, s5675);
    s5677 = _mm256_unpackhi_pd(s5674, s5675);
    s5678 = *((a3738 + 16));
    s5679 = *((a3738 + 17));
    s5680 = _mm256_permute2f128_pd(s5678, s5679, (0) | ((2) << 4));
    s5681 = _mm256_permute2f128_pd(s5678, s5679, (1) | ((3) << 4));
    s5682 = _mm256_unpacklo_pd(s5680, s5681);
    s5683 = _mm256_unpackhi_pd(s5680, s5681);
    t5735 = _mm256_add_pd(s5676, s5682);
    t5736 = _mm256_add_pd(s5677, s5683);
    t5737 = _mm256_add_pd(s5670, t5735);
    t5738 = _mm256_add_pd(s5671, t5736);
    t5739 = _mm256_sub_pd(s5670, _mm256_mul_pd(_mm_vbroadcast_sd(&(C22)), t5735));
    t5740 = _mm256_sub_pd(s5671, _mm256_mul_pd(_mm_vbroadcast_sd(&(C22)), t5736));
    t5741 = _mm256_mul_pd(_mm_vbroadcast_sd(&(C23)), _mm256_sub_pd(s5677, s5683));
    t5742 = _mm256_mul_pd(_mm_vbroadcast_sd(&(C23)), _mm256_sub_pd(s5676, s5682));
    ...
}
```



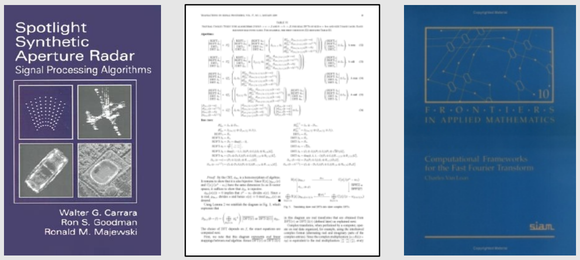

**performance**

+


**QED.**

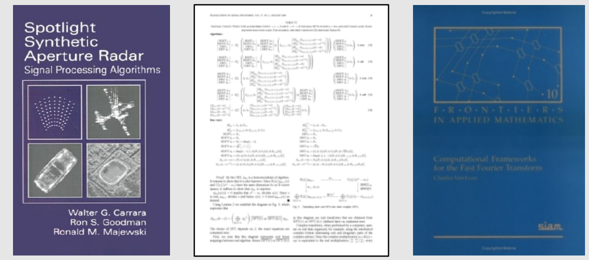
# The Spiral System

## *Traditionally*




High performance library
optimized for given platform

## *Spiral Approach*



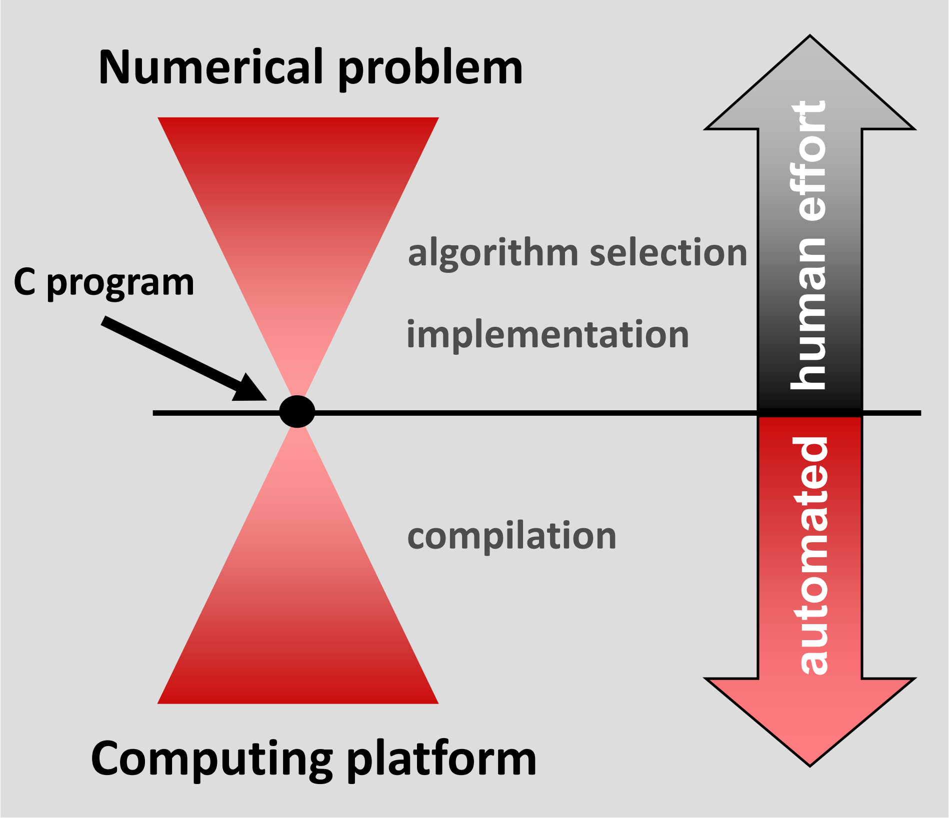

***Spiral***

High performance library
optimized for given platform

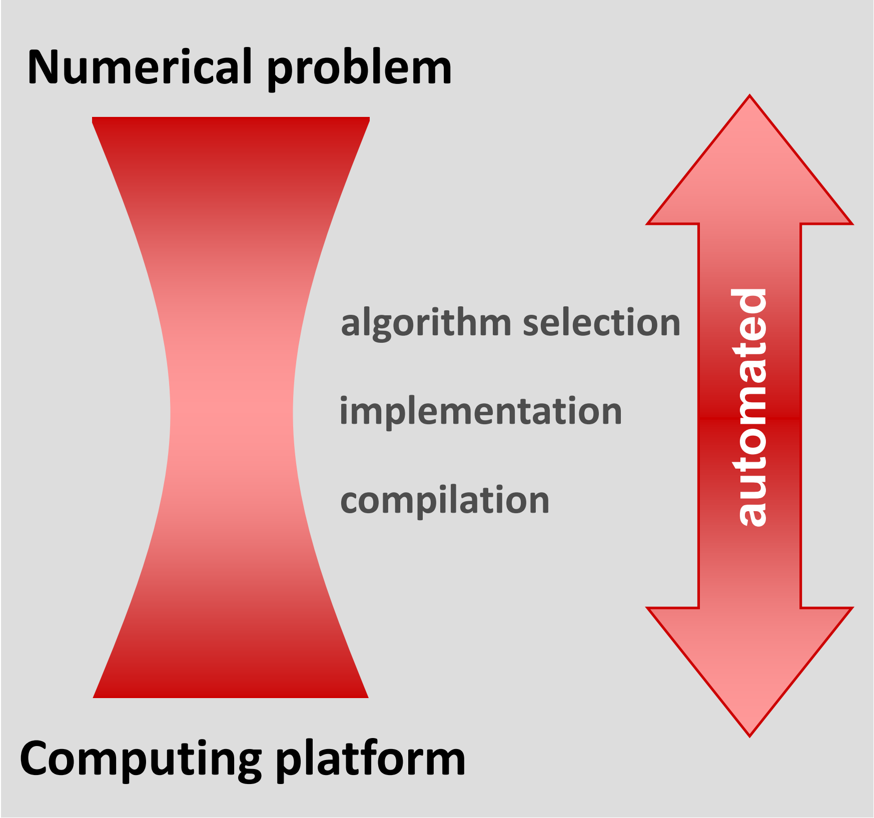
***Comparable performance***

# Vision Behind Spiral

*Current*




*Future*



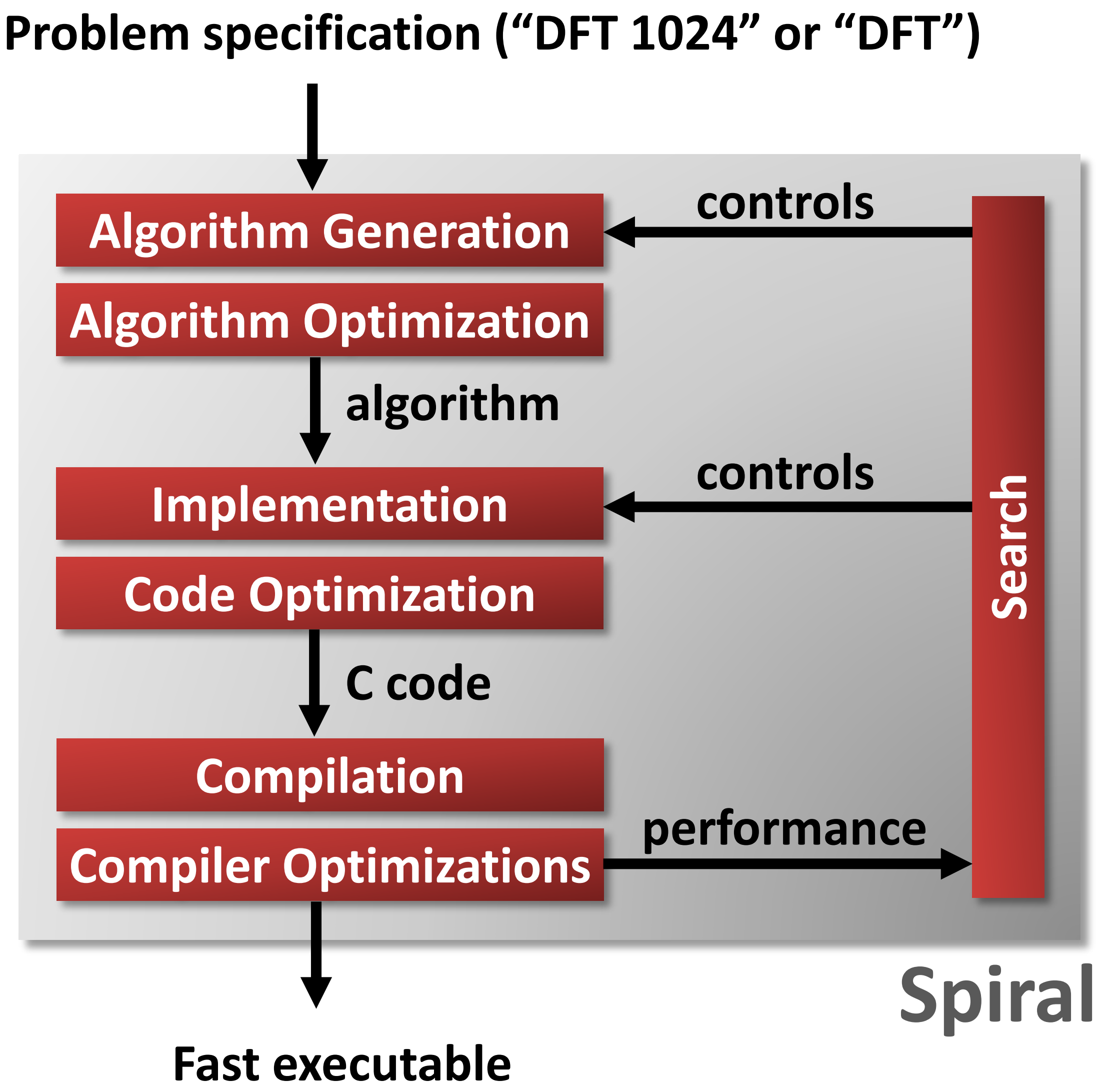

- **C code a singularity: Compiler has no access to high level information**
- **Challenge: conquer the high abstraction level for complete automation**

# How Spiral Works

***Complete automation*** of the implementation and optimization task

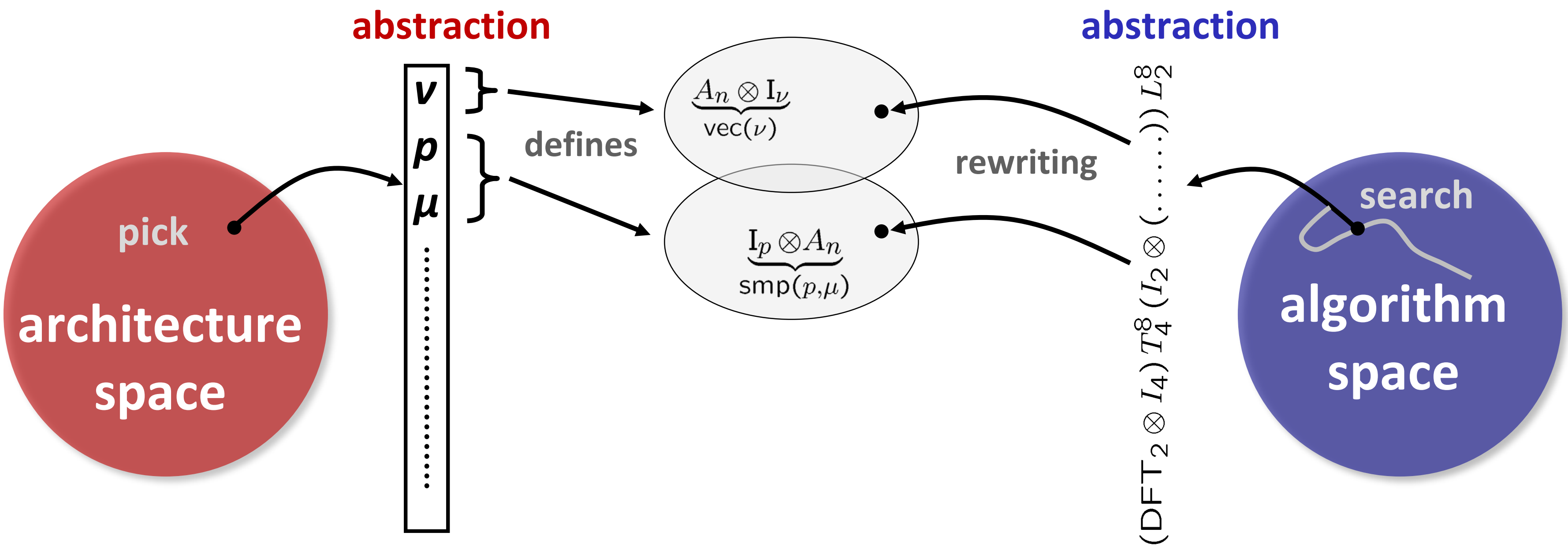
**Basic ideas:**

- ***Declarative representation*** of algorithms
- ***Rewriting systems*** to generate and optimize algorithms at a high level of abstraction



# Spiral's Domain-Specific Program Synthesis

**Model:** common abstraction
= spaces of matching formulas




Architectural parameter:
Vector length,
#processors, …

optimization

Kernel:
problem size,
algorithm choice

# Example 1: SAR for Cell BE


**Result**

**Same performance, 1/10th human effort, non-expert user**

**Key ideas**

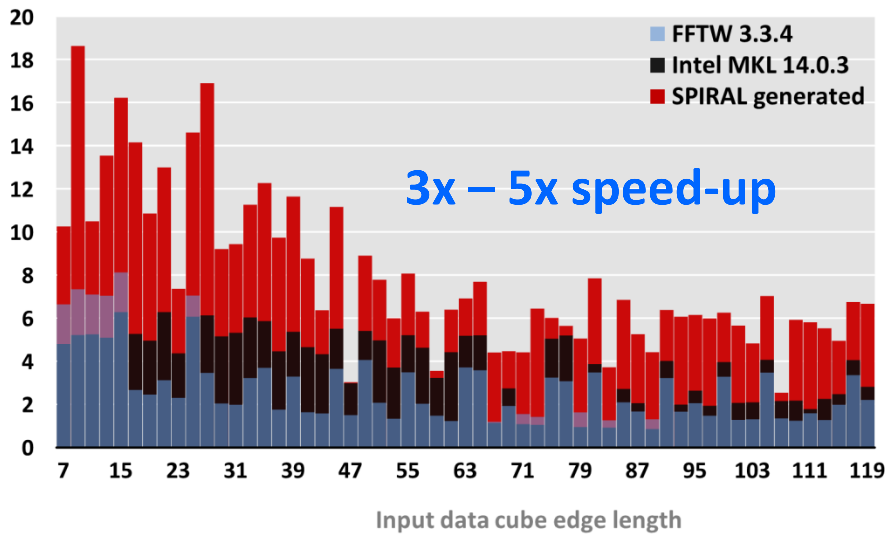
**restrict domain, use mathematics, performance portability**

D. McFarlin, F. Franchetti, M. Püschel, and J. M. F. Moura: **High Performance Synthetic Aperture Radar Image Formation On Commodity Multicore Architectures.** in Proceedings SPIE, 2009.

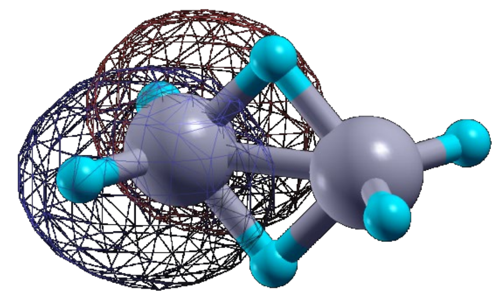
# Example 2: Density Functional Theory

**Performance of 2x2x2 Upsampling on Haswell**
***3.5 GHz, AVX, double precision, interleaved input, single core***

Performance [Pseudo Gflop/s]

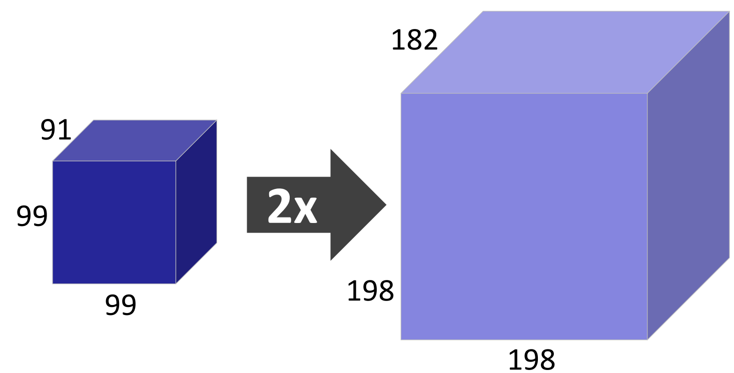



ONETEP



quantum-mechanical calculations based on density-functional theory

## Core operation:
## FFT-based 3D 2x2x2 upsampling




## Unusual requirements:

- Odd FFT sizes
- Small sizes
- Rectangular box

**ONETEP = Order-*N* Electronic Total Energy Package**

P. D. Haynes, C.-K. Skylaris, A. A. Mostofi and M. C. Payne, **"ONETEP: linear-scaling density-functional theory with plane waves,"** Psi-k Newsletter 72, 78-91 (December 2005)

T. Popovici, F. Russell, K. Wilkinson, C-K. Skylaris, P. H. J. Kelly, F. Franchetti, **"Generating Optimized Fourier Interpolation Routines for Density Functional Theory Using SPIRAL,"** 29th International Parallel & Distributed Processing Symposium (IPDPS), 2015.

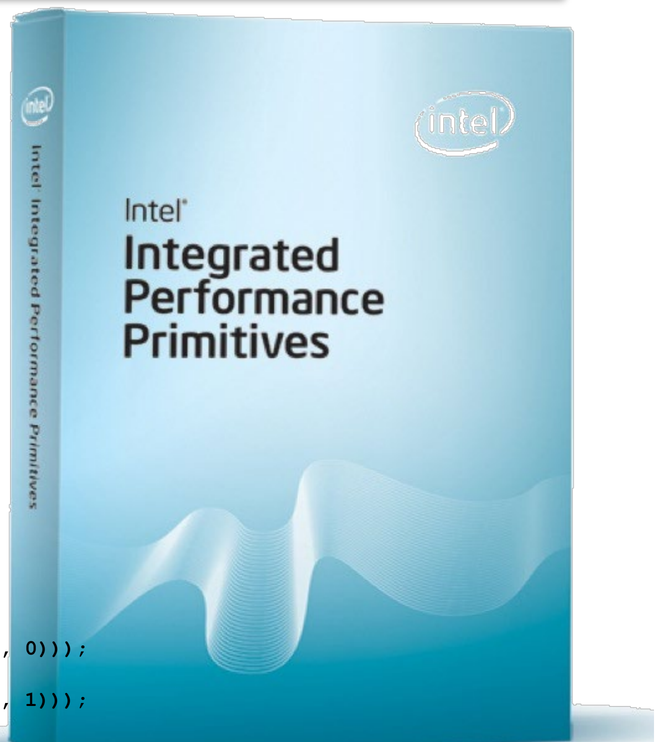
# Example 3: Synthesis of Production Code

```
...
s3013 = _mm_loadl_pi(a772, ((float  *) X));
s3014 = _mm_loadh_pi(_mm_loadl_pi(a772, ((float  *) (X + 2))), ((float  *) (X + 6)));
```

## *Spiral-Synthesized code in Intel's Library IPP 6 and 7*

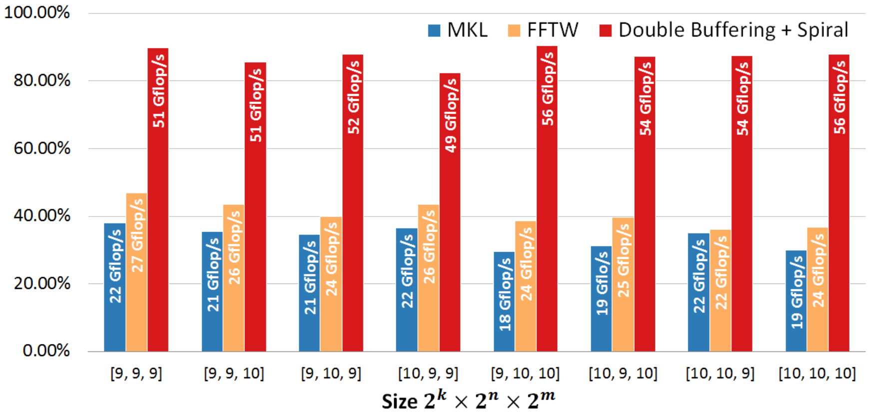
- IPP = Intel's performance primitives, part of Intel C++ Compiler suite
- Generated: 3984 C functions (signal processing) = 1M lines of code
- Full parallelism support
- Computer-generated code: Faster than what was achievable by hand

```
s3017 = _mm_loadl_pi(a772, ((float  *) (X + 14)));
s3018 = _mm_loadh_pi(_mm_loadl_pi(a772, ((float  *) (X + 24))), ((float  *) (X + 20)));
s3019 = _mm_loadl_pi(a772, ((float  *) (X + 8)));
s3020 = _mm_loadh_pi(_mm_loadl_pi(a772, ((float  *) (X + 10))), ((float  *) (X + 4)));
s3021 = _mm_loadl_pi(a772, ((float  *) (X + 12)));
s3022 = _mm_shuffle_ps(s3014, s3015, _MM_SHUFFLE(2, 0, 2, 0));
s3023 = _mm_shuffle_ps(s3014, s3015, _MM_SHUFFLE(3, 1, 3, 1));
s3024 = _mm_shuffle_ps(s3016, s3017, _MM_SHUFFLE(2, 0, 2, 0));
s3025 = _mm_shuffle_ps(s3016, s3017, _MM_SHUFFLE(3, 1, 3, 1));
s3026 = _mm_shuffle_ps(s3018, s3019, _MM_SHUFFLE(2, 0, 2, 0));
s3027 = _mm_shuffle_ps(s3018, s3019, _MM_SHUFFLE(3, 1, 3, 1));
s3028 = _mm_shuffle_ps(s3020, s3021, _MM_SHUFFLE(2, 0, 2, 0));
s3029 = _mm_shuffle_ps(s3020, s3021, _MM_SHUFFLE(3, 1, 3, 1));
...
t3794 = _mm_add_ps(s3042, s3043);
t3795 = _mm_add_ps(s3038, t3793);
t3796 = _mm_add_ps(s3041, t3794);
t3797 = _mm_sub_ps(s3038, _mm_mul_ps(_mm_set1_ps(0.5), t3793));
t3798 = _mm_sub_ps(s3041, _mm_mul_ps(_mm_set1_ps(0.5), t3794));
s3044 = _mm_mul_ps(_mm_set1_ps(0.8660254037844386), _mm_sub_ps(s3042, s3043));
s3045 = _mm_mul_ps(_mm_set1_ps(0.8660254037844386), _mm_sub_ps(s3039, s3040));
t3799 = _mm_add_ps(t3797, s3044);
t3800 = _mm_sub_ps(t3798, s3045);
t3801 = _mm_sub_ps(t3797, s3044);
t3802 = _mm_add_ps(t3798, s3045);
a773 = _mm_mul_ps(_mm_set_ps(0, 0, 0, 1), _mm_shuffle_ps(s3013, a772, _MM_SHUFFLE(2, 0, 2, 0)));
t3803 = _mm_add_ps(a773, _mm_mul_ps(_mm_set_ps(0, 0, 0, 1), t3795));
a774 = _mm_mul_ps(_mm_set_ps(0, 0, 0, 1), _mm_shuffle_ps(s3013, a772, _MM_SHUFFLE(3, 1, 3, 1)));
t3804 = _mm_add_ps(a774, _mm_mul_ps(_mm_set_ps(0, 0, 0, 1), t3796));
t3805 = _mm_add_ps(a773, _mm_add_ps(_mm_mul_ps(_mm_set_ps(0.28757036473700154, 0.30046260628866578, (-0.28757036473700154), (-0.0833333333333333
t3806 = _mm_add_ps(a774, _mm_add_ps(_mm_mul_ps(_mm_set_ps(0.08706930057606789, 0, 0.087069300576068001, 0), t3795), _mm_mul_ps(_mm_set_ps(0.2875
s3046 = _mm_sub_ps(_mm_mul_ps(_mm_set_ps((-0.25624767158293649), 0.25826039031174486, (-0.30023863596633249), 0.07590298603719387 9), t3799), _mm
s3047 = _mm_add_ps(_mm_mul_ps(_mm_set_ps(0.15689139105158462, (-0.15355568557954136), (-0.011599105605768193), 0.2907172414708409 9), t3799), _mm
```



# Selected Results: FFTs and Spectral Algorithms

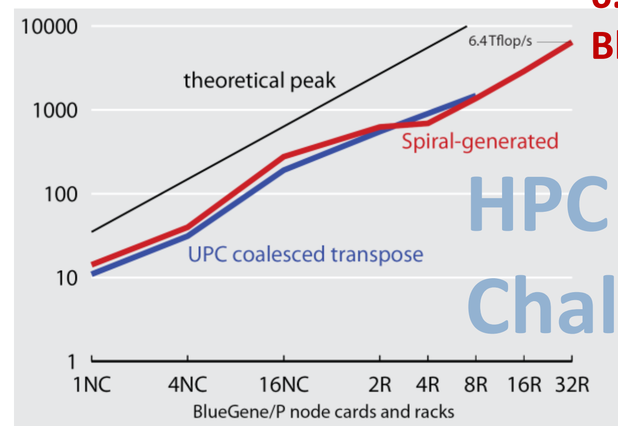
**3D FFT performance on Intel Kaby Lake 7700K**
4.5 GHz, 4/8 cores/threads, double-precision, AVX
[% of achievable peak performance given STREAM bandwidth]




**FFT on Multicore**

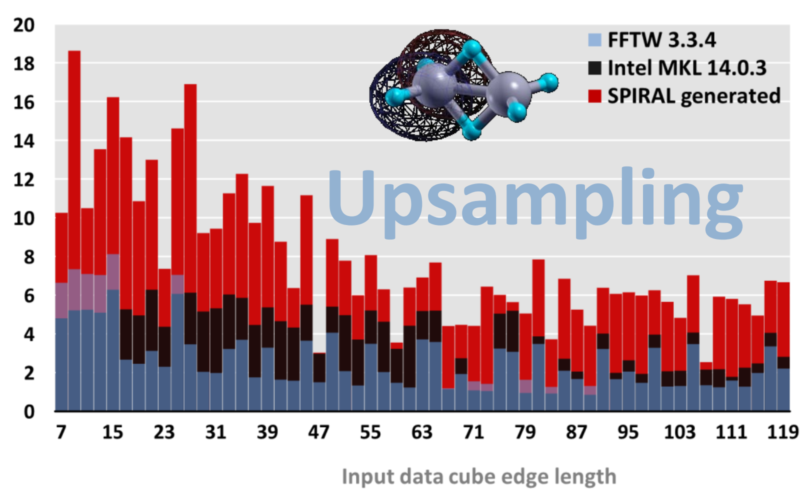
**Global FFT (1D FFT, HPC Challenge)**
performance [Gflop/s]

**6.4 Tflop/s on BlueGene/P**

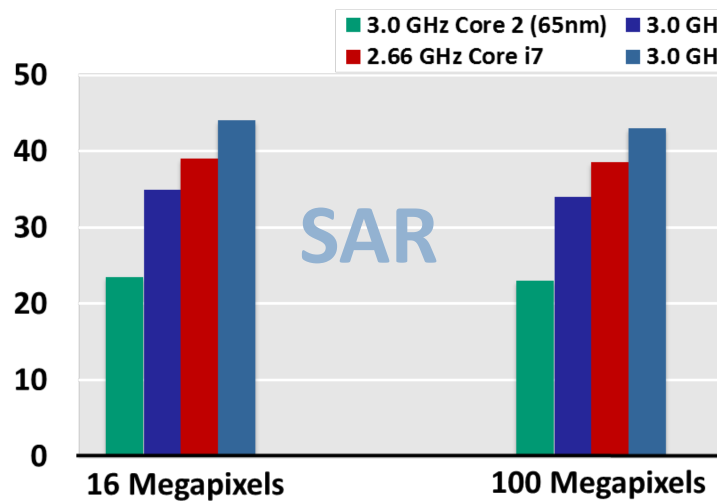


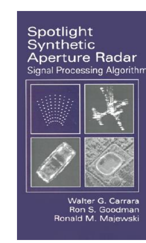

BlueGene/P at Argonne National Laboratory
128k cores (quad-core CPUs) at 850 MHz

**Performance of 2x2x2 Upsampling on Haswell**
***3.5 GHz, AVX, double precision, interleaved input, single core***
Performance [Pseudo Gflop/s]




**PFA SAR Image Formation on Intel platforms**
performance [Gflop/s]



# Current Work: FFTX and SpectralPACK

| Numerical Linear Algebra | | Spectral Algorithms |
|---|---|---|
| **LAPACK**<br>LU factorization<br>Eigensolves<br>SVD<br>… | ≈ | **SpectralPACK**<br>Convolution<br>Correlation<br>Upsampling<br>Poisson solver<br>… |
| **BLAS**<br>BLAS-1<br>BLAS-2<br>BLAS-3 | | **FFTX**<br>DFT, RDFT<br>1D, 2D, 3D,…<br>batch |

**Define the LAPACK equivalent for spectral algorithms**

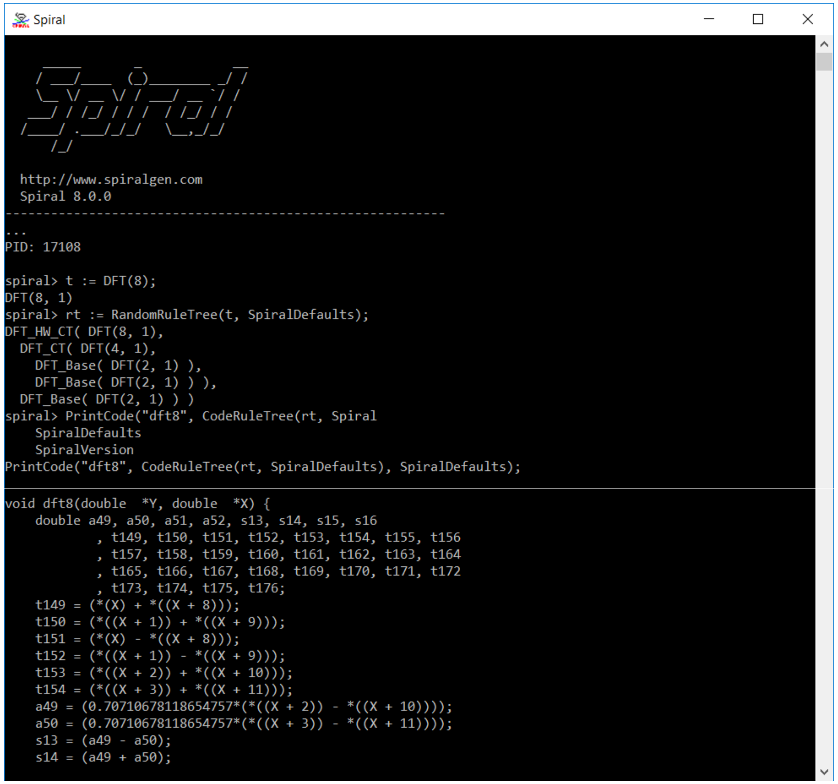
- **Define FFTX as the BLAS equivalent**
  provide user FFT functionality as well as algorithm building blocks
- **Define class of numerical algorithms to be supported by SpectralPACK**
  PDE solver classes (Green's function, sparse in normal/k space,…), signal processing,…
- **Library front-end, code generation and vendor library back-end**
  mirror concepts from FFTX layer

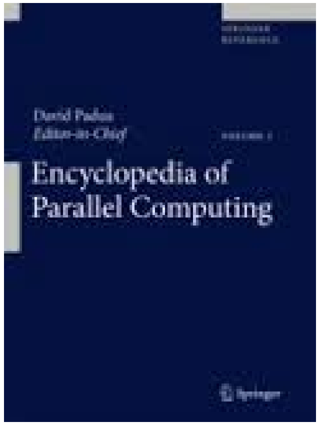
***Spiral provides backend code generation and autotuning***

# SPIRAL 8.0: Open Source

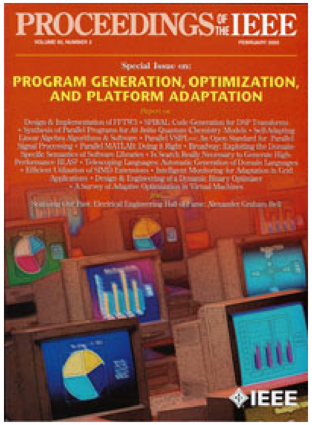
- **Open Source SPIRAL available**
  - non-viral license (BSD)
  - Initial version, effort ongoing to open source whole system
  - Open sourced under DARPA PERFECT
  - Commercial support via SpiralGen, Inc.
- **Developed over 20 years**

  Funding: DARPA (OPAL, DESA, HACMS, PERFECT, BRASS), NSF, ONR, DoD HPC, JPL, DOE, CMU SEI, Intel, Nvidia, Mercury








**www.spiral.net**

F. Franchetti, T. M. Low, D. T. Popovici, R. M. Veras, D. G. Spampinato, J. R. Johnson, M. Püschel, J. C. Hoe, J. M. F. Moura: **SPIRAL: Extreme Performance Portability,** Proceedings of the IEEE, Vol. 106, No. 11, 2018. Special Issue on *From High Level Specification to High Performance Code*

# Organization

- **SPL: Problem and algorithm specification**
- Σ-SPL: Automating high level optimization
- Rewriting: Formal parallelization
- Rewriting: Vectorization
- Verification
- Spiral as FFTX backend
- Summary

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Special Issue on *From High Level Specification to High Performance Code*

# What is a (Linear) Transform?

- Mathematically: Matrix-vector multiplication

$$x \mapsto y = T \cdot x$$

input vector (signal)

output vector (signal)

transform = matrix

- Example: Discrete Fourier transform (DFT)

$$\mathrm{DFT}_n \;=\; [e^{-2k\ell\pi i/n}]_{0\leq k,\ell<n}$$

# Transform Algorithms: Example 4-point FFT

**Cooley/Tukey fast Fourier transform (FFT):**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & j \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Fourier transform | Diagonal matrix (twiddles)

$$\mathbf{DFT}_4 \rightarrow (\mathbf{DFT}_2 \otimes \mathrm{I}_2)\,\mathrm{T}^4_2\,(\mathrm{I}_2 \otimes \mathbf{DFT}_2)\,\mathrm{L}^4_2$$

Kronecker product | Identity | Permutation

- Algorithms are divide-and-conquer: Breakdown rules
- Mathematical, declarative representation: SPL (signal processing language)
- SPL describes the structure of the dataflow

# Examples: Transforms

$$\mathbf{DCT\text{-}2}_n = \left[\cos(k(2\ell+1)\pi/2n)\right]_{0\le k,\ell<n},$$

$$\mathbf{DCT\text{-}3}_n = \mathbf{DCT\text{-}2}_n^T \quad \text{(transpose)},$$

$$\mathbf{DCT\text{-}4}_n = \left[\cos((2k+1)(2\ell+1)\pi/4n)\right]_{0\le k,\ell<n},$$

$$\mathbf{IMDCT}_n = \left[\cos((2k+1)(2\ell+1+n)\pi/4n)\right]_{0\le k<2n,0\le\ell<n},$$

$$\mathbf{RDFT}_n = [r_{k\ell}]_{0\le k,\ell<n}, \quad r_{k\ell} = \begin{cases} \cos\frac{2\pi k\ell}{n}, & k \le \lfloor\frac{n}{2}\rfloor \\ -\sin\frac{2\pi k\ell}{n}, & k > \lfloor\frac{n}{2}\rfloor \end{cases},$$

$$\mathbf{WHT}_n = \begin{bmatrix} \mathbf{WHT}_{n/2} & \mathbf{WHT}_{n/2} \\ \mathbf{WHT}_{n/2} & -\mathbf{WHT}_{n/2} \end{bmatrix}, \quad \mathbf{WHT}_2 = \mathbf{DFT}_2,$$

$$\mathbf{DHT} = \left[\cos(2k\ell\pi/n) + \sin(2k\ell\pi/n)\right]_{0\le k,\ell<n}.$$

# Examples: Breakdown Rules (currently ≈220)

$$\begin{aligned}
\mathbf{DFT}_n &\rightarrow (\mathbf{DFT}_k \otimes \mathrm{I}_m)\, \mathsf{T}^n_m (\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathsf{L}^n_k, \quad n = km \\
\mathbf{DFT}_n &\rightarrow P_n(\mathbf{DFT}_k \otimes \mathbf{DFT}_m) Q_n, \quad n = km,\ \mathsf{gcd}(k,m) = 1 \\
\mathbf{DFT}_p &\rightarrow R_p^T(\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) D_p (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) R_p, \quad p \text{ prime} \\
\mathbf{DCT\text{-}3}_n &\rightarrow (\mathrm{I}_m \oplus \mathrm{J}_m)\, \mathsf{L}^n_m (\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4)) \\
&\qquad \cdot(\mathrm{F}_2 \otimes \mathrm{I}_m) \begin{bmatrix} \mathrm{I}_m & 0 \oplus -\mathrm{J}_{m-1} \\ & \frac{1}{\sqrt{2}}(\mathrm{I}_1 \oplus 2\,\mathrm{I}_m) \end{bmatrix}, \quad n = 2m \\
\mathbf{DCT\text{-}4}_n &\rightarrow S_n \mathbf{DCT\text{-}2}_n\, \mathsf{diag}_{0 \le k < n}(1/(2\cos((2k+1)\pi/4n))) \\
\mathbf{IMDCT}_{2m} &\rightarrow (\mathrm{J}_m \oplus \mathrm{I}_m \oplus \mathrm{I}_m \oplus \mathrm{J}_m) \left( \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \mathrm{I}_m \right) \oplus \left( \begin{bmatrix} -1 \\ -1 \end{bmatrix} \otimes \mathrm{I}_m \right) \right) \mathrm{J}_{2m}\, \mathbf{DCT\text{-}4}_{2m} \\
\mathbf{WHT}_{2^k} &\rightarrow \prod_{i=1}^{t} (\mathrm{I}_{2^{k_1+\cdots+k_{i-1}}} \otimes \mathbf{WHT}_{2^{k_i}} \otimes \mathrm{I}_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t \\
\mathbf{DFT}_2 &\rightarrow \mathrm{F}_2 \\
\mathbf{DCT\text{-}2}_2 &\rightarrow \mathsf{diag}(1, 1/\sqrt{2})\, \mathrm{F}_2 \\
\mathbf{DCT\text{-}4}_2 &\rightarrow \mathrm{J}_2\, \mathrm{R}_{13\pi/8}
\end{aligned}$$

*Combining these rules yields many algorithms for every given transform*

# Breakdown Rules (>200 for >50 Transforms)

$$\mathbf{DFT}_n \to P^{\top}_{k/2,2m}\left(\mathbf{DFT}_{2m} \oplus \left(I_{k/2-1} \otimes_i C_{2m}\,\mathbf{rDFT}_{2m}(i/k)\right)\right)\left(\mathbf{RDFT}'_k \otimes I_m\right), \quad k \text{ even},$$

$$\begin{vmatrix}\mathbf{RDFT}_n\\ \mathbf{RDFT}'_n\\ \mathbf{DHT}_n\\ \mathbf{DHT}'_n\end{vmatrix} \to (P^{\top}_{k/2,m} \otimes I_2)\left(\begin{vmatrix}\mathbf{RDFT}_{2m}\\ \mathbf{RDFT}'_{2m}\\ \mathbf{DHT}_{2m}\\ \mathbf{DHT}'_{2m}\end{vmatrix} \oplus \left(I_{k/2-1} \otimes_i D_{2m}\begin{vmatrix}\mathbf{rDFT}_{2m}(i/k)\\ \mathbf{rDFT}_{2m}(i/k)\\ \mathbf{rDHT}_{2m}(i/k)\\ \mathbf{rDHT}_{2m}(i/k)\end{vmatrix}\right)\right)\left(\begin{vmatrix}\mathbf{RDFT}'_k\\ \mathbf{RDFT}'_k\\ \mathbf{DHT}'_k\\ \mathbf{DHT}'_k\end{vmatrix} \otimes I_m\right), \quad k \text{ even},$$

$$\begin{vmatrix}\mathbf{rDFT}_{2n}(u)\\ \mathbf{rDHT}_{2n}(u)\end{vmatrix} \to L^{2n}_m\left(I_k \otimes_i \begin{vmatrix}\mathbf{rDFT}_{2m}((i+u)/k)\\ \mathbf{rDHT}_{2m}((i+u)/k)\end{vmatrix}\right)\left(\begin{vmatrix}\mathbf{rDFT}_{2k}(u)\\ \mathbf{rDHT}_{2k}(u)\end{vmatrix} \otimes I_m\right),$$

$$\mathbf{RDFT\text{-}3}_n \to (Q^{\top}_{k/2,m} \otimes I_2)\,(I_k \otimes_i \mathbf{rDFT}_{2m})(i+1/2)/k))\,(\mathbf{RDFT\text{-}3}_k \otimes I_m), \quad k \text{ even},$$

$$\mathbf{DCT\text{-}2}_n \to P^{\top}_{k/2,2m}\left(\mathbf{DCT\text{-}2}_{2m}\,K^{2m}_2 \oplus \left(I_{k/2-1} \otimes N_{2m}\,\mathbf{RDFT\text{-}3}^{\top}_{2m}\right)\right)B_n(L^{n/2}_{k/2} \otimes I_2)(I_m \otimes \mathbf{RDFT}'_k)Q_{m/2,k},$$

$$\mathbf{DCT\text{-}3}_n \to \mathbf{DCT\text{-}2}^{\top}_n,$$

$$\mathbf{DCT\text{-}4}_n \to Q^{\top}_{k/2,2m}\left(I_{k/2} \otimes N_{2m}\,\mathbf{RDFT\text{-}3}^{\top}_{2m}\right)B'_n(L^{n/2}_{k/2} \otimes I_2)(I_m \otimes \mathbf{RDFT\text{-}3}_k)Q_{m/2,k}.$$

$$\mathbf{DFT}_n \to (\mathbf{DFT}_k \otimes \mathrm{I}_m)\,\mathrm{T}^n_m(\mathrm{I}_k \otimes \mathbf{DFT}_m)\,\mathrm{L}^n_k, \quad n = km$$

$$\mathbf{DFT}_n \to P_n(\mathbf{DFT}_k \otimes \mathbf{DFT}_m)Q_n, \quad n = km,\ \gcd(k,m) = 1$$

$$\mathbf{DFT}_p \to R^T_p(\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1})D_p(\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1})R_p, \quad p \text{ prime}$$

$$\mathbf{DCT\text{-}3}_n \to (\mathrm{I}_m \oplus \mathrm{J}_m)\,\mathrm{L}^n_m(\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4)) \cdot (\mathrm{F}_2 \otimes \mathrm{I}_m)\begin{bmatrix}\mathrm{I}_m & 0 \oplus -\mathrm{J}_{m-1}\\ & \frac{1}{\sqrt{2}}(\mathrm{I}_1 \oplus 2\,\mathrm{I}_m)\end{bmatrix}, \quad n = 2m$$

$$\mathbf{DCT\text{-}4}_n \to S_n\mathbf{DCT\text{-}2}_n\,\mathrm{diag}_{0\le k<n}(1/(2\cos((2k+1)\pi/4n)))$$

$$\mathbf{IMDCT}_{2m} \to (\mathrm{J}_m \oplus \mathrm{I}_m \oplus \mathrm{I}_m \oplus \mathrm{J}_m)\left(\left(\begin{bmatrix}1\\-1\end{bmatrix} \otimes \mathrm{I}_m\right) \oplus \left(\begin{bmatrix}-1\\-1\end{bmatrix} \otimes \mathrm{I}_m\right)\right)\mathrm{J}_{2m}\,\mathbf{DCT\text{-}4}_{2m}$$

$$\mathbf{WHT}_{2^k} \to \prod_{i=1}^{t}(\mathrm{I}_{2^{k_1+\cdots+k_{i-1}}} \otimes \mathbf{WHT}_{2^{k_i}} \otimes \mathrm{I}_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t$$

$$\mathbf{DFT}_2 \to \mathrm{F}_2$$

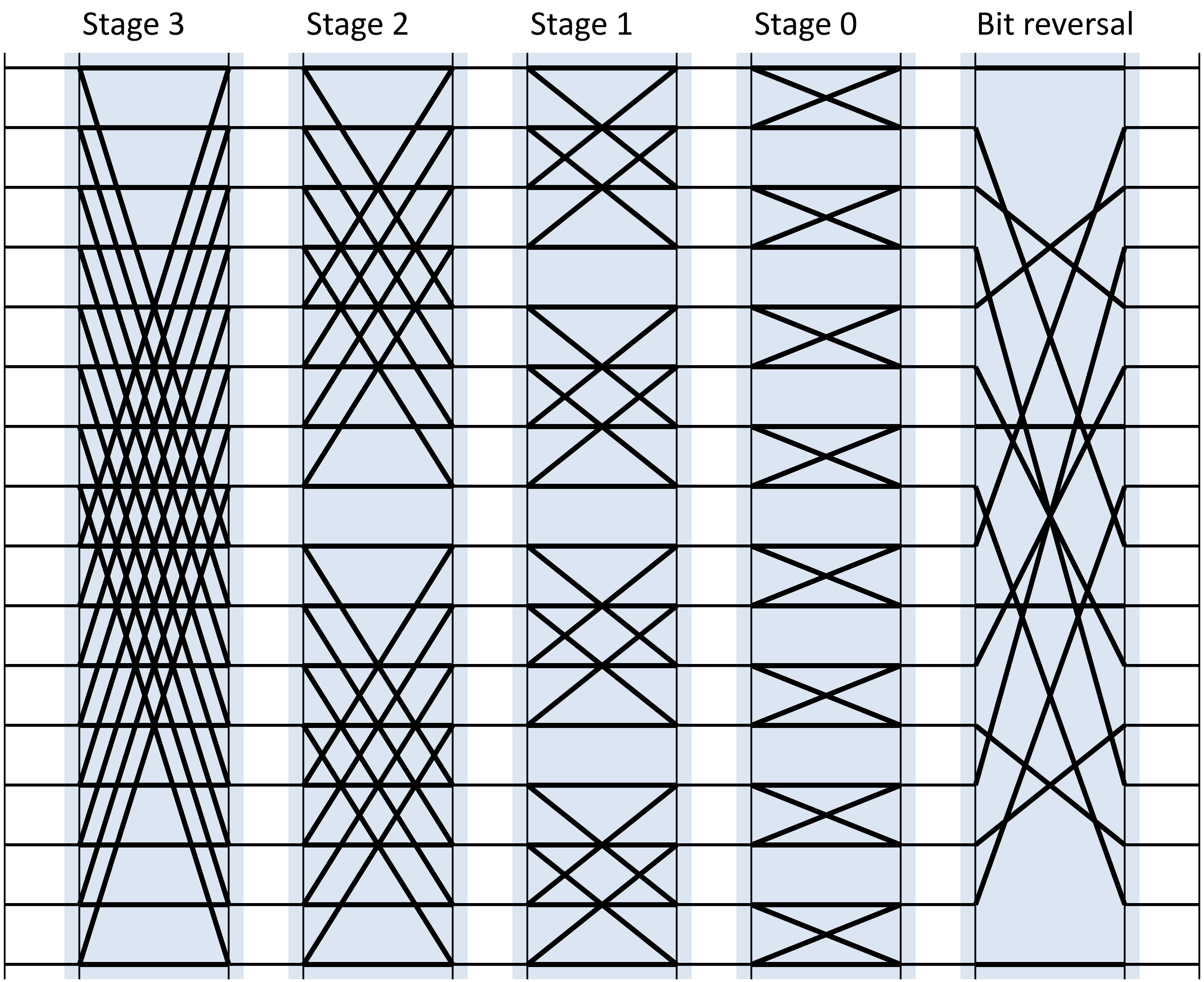
$$\mathbf{DCT\text{-}2}_2 \to \mathrm{diag}(1, 1/\sqrt{2})\,\mathrm{F}_2$$

$$\mathbf{DCT\text{-}4}_2 \to \mathrm{J}_2\,\mathrm{R}_{13\pi/8}$$

- *"Teaches" Spiral algorithm knowledge*
- *Combining these rules yields many algorithms for every given transform*

# Example FFT: Iterative FFT Algorithm

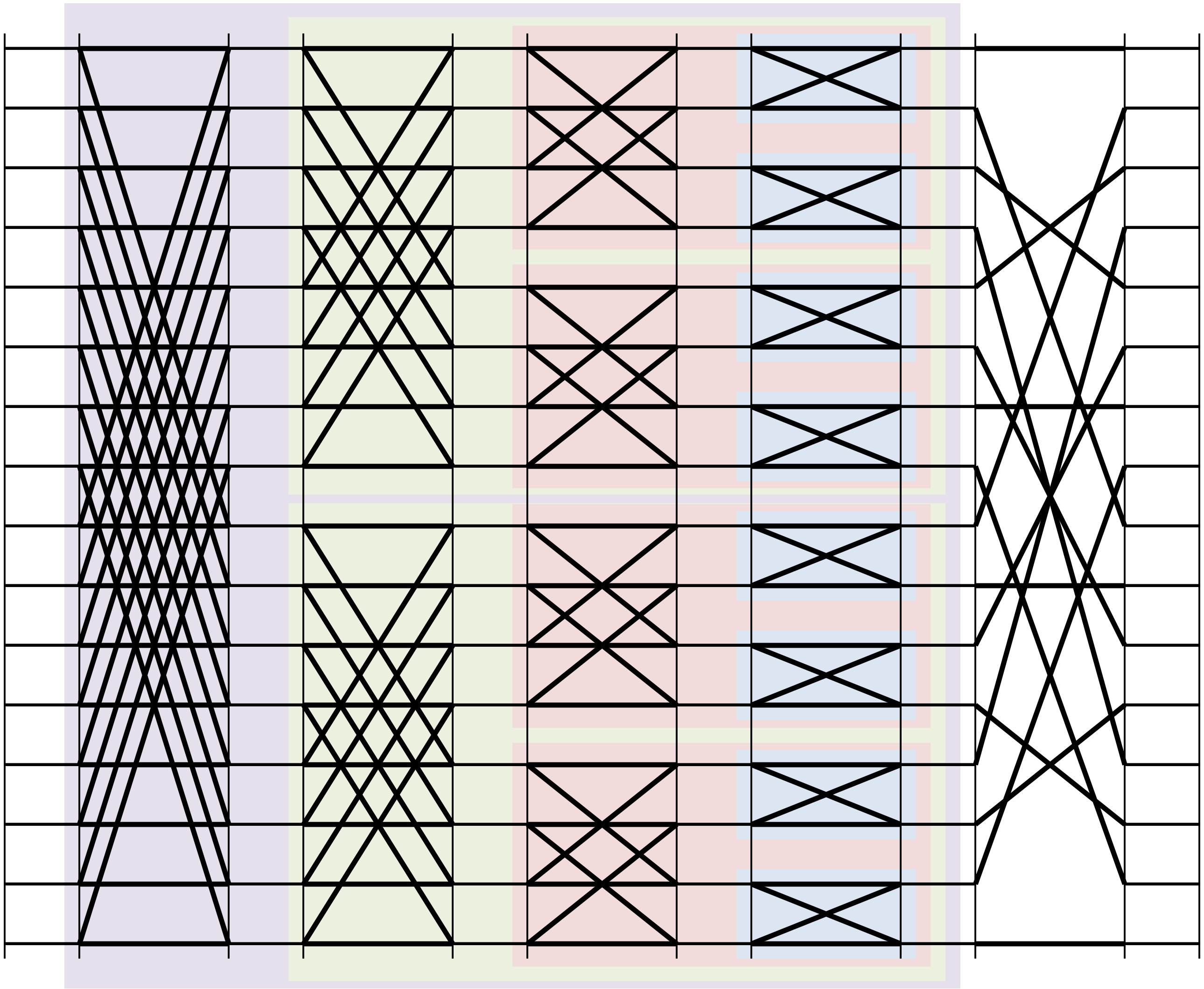
$$\mathrm{DFT}_{r^k} = \left( \prod_{i=0}^{k-1} (I_{r^i} \otimes \mathrm{DFT}_r \otimes I_{r^{k-i-1}}) D_i^{r^k} \right) R_r^{r^k}$$




$$\left((I_1 \otimes \mathrm{DFT}_2 \otimes I_8) D_0^{16}\right) \left((I_2 \otimes \mathrm{DFT}_2 \otimes I_4) D_1^{16}\right) \left((I_4 \otimes \mathrm{DFT}_2 \otimes I_2) D_2^{16}\right) \left((I_8 \otimes \mathrm{DFT}_2 \otimes I_1) D_3^{16}\right) R_2^{16}$$

# Example FFT: Recursive FFT Algorithm

$$\mathrm{DFT}_{km} = (\mathrm{DFT}_k \otimes I_m) T^n_m (I_k \otimes \mathrm{DFT}_m) L^n_k$$



$$(\mathrm{DFT}_2 \otimes I_8) T^{16}_8 \Big( I_2 \otimes \Big( (\mathrm{DFT}_2 \otimes I_4) T^8_4 \big( I_2 \otimes \big( (\mathrm{DFT}_2 \otimes I_2) T^4_2 (I_2 \otimes \mathrm{DFT}_2) L^4_2 \big) \big) L^8_2 \Big) \Big) L^{16}_2$$

# SPL Compiler

| SPL construct | code |
| --- | --- |
| $y = (A_n B_n)x$ | `t[0:1:n-1] = B(x[0:1:n-1]);` `y[0:1:n-1] = A(t[0:1:n-1];)` |
| $y = (I_m \otimes A_n)x$ | `for (i=0;i<m;i++)` `y[i*n:1:i*n+n-1] =` `A(x[i*n:1:i*n+n-1])` |
| $y = (A_m \otimes I_n)x$ | `for (i=0;i<m;i++)` `y[i:n:i+m-1] =` `A(x[i:n:i+m-1]);` |
| $y = \left(\bigoplus_{i=0}^{m-1} A_n^i\right) x$ | `for (i=0;i<m;i++)` `y[i*n:1:i*n+n-1] =` `A(i, x[i*n:1:i*n+n-1]);` |
| $y = D_{m,n} x$ | `for (i=0;i<m*n;i++)` `y[i] = Dmn[i]*x[i];` |
| $y = L_m^{mn} x$ | `for (i=0;i<m;i++)` `for (j=0;j<n;j++)` `y[i+m*j]=x[n*i+j];` |

**Example: tensor product**

$$\mathrm{I}_m \otimes A_n = \begin{bmatrix} A_n & & \\ & \ddots & \\ & & A_n \end{bmatrix}$$

***Works well for basic blocks. Loops and parallelization: next***

# Organization

- SPL: Problem and algorithm specification
- **Σ-SPL: Automating high level optimization**
- Rewriting: Formal parallelization
- Rewriting: Vectorization
- Verification
- Spiral as FFTX backend
- Summary

F. Franchetti, Y. Voronenko, M. Püschel: **Loop Merging for Signal Transforms**
Proceedings Programming Language Design and Implementation (PLDI) 2005, pages 315-326.

# Problem: Fusing Permutations and Loops

$$y = (I_4 \otimes F_2) L^8_4 \, x$$

direct mapping

Two passes over the working set
Complex index computation

```
void sub(double *y, double *x) {
  double t[8];
  for (int i=0; i<=7; i++)
     t[(i/4)+2*(i%4)] = x[i];
  for (int i=0; i<4; i++){
     y[2*i] = t[2*i] + t[2*i+1];
     y[2*i+1] = t[2*i] - t[2*i+1];
  }
}
```

C compiler cannot do this

hardcoded special case

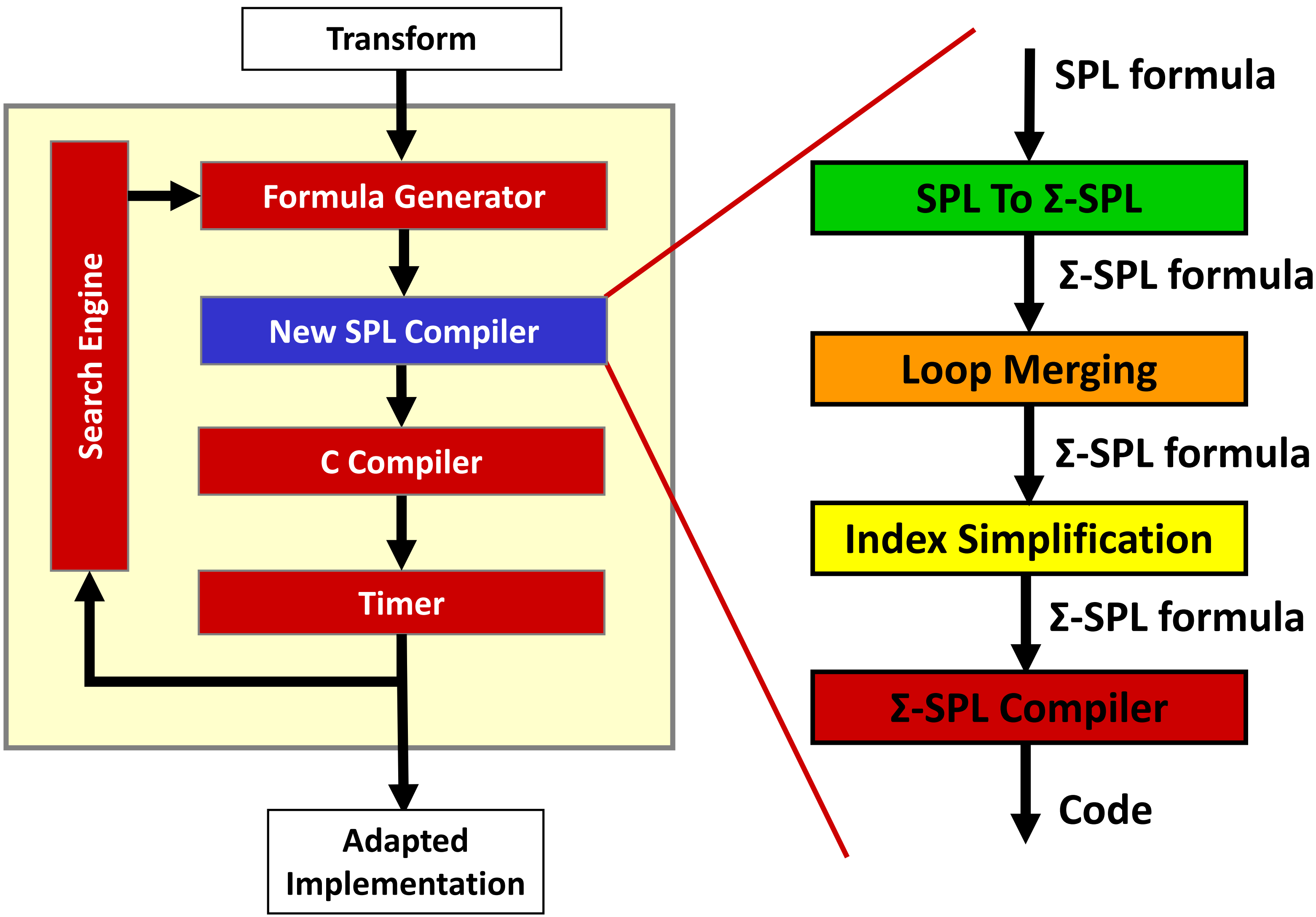
One pass over the working set
Simple index computation

```
void sub(double *y, double *x) {
   for (int j=0; j<=3; j++){
      y[2*j] = x[j] + x[j+4];
      y[2*j+1] = x[j] - x[j+4];
   }
}
```

State-of-the-art
SPIRAL: Hardcoded with templates
FFTW: Hardcoded in the infrastructure

How does hardcoding scale?

# New Approach for Loop Merging



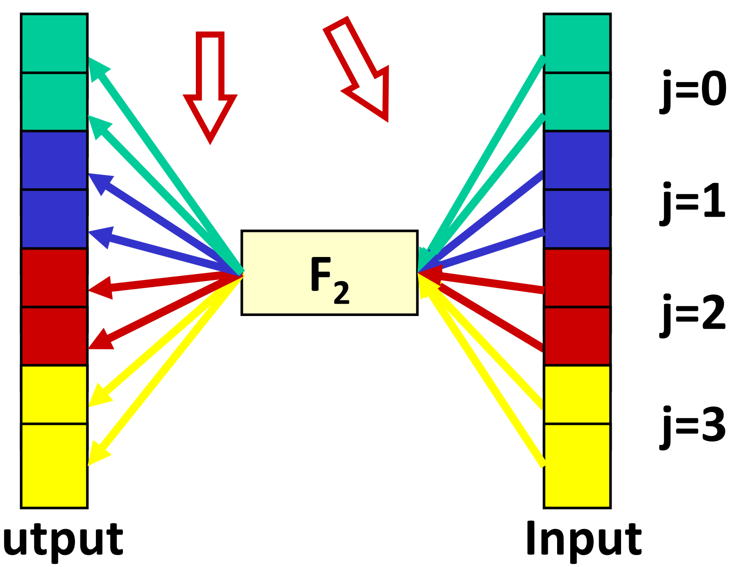
# Σ-SPL

- **Four central constructs: S, G, S, Perm**
  - Σ (sum) – makes loops explicit
  - $G_f$ (gather) – reads data using the index mapping $f$
  - $S_f$ (scatter) – writes data using the index mapping $f$
  - $Perm_f$ – permutes data using the index mapping $f$

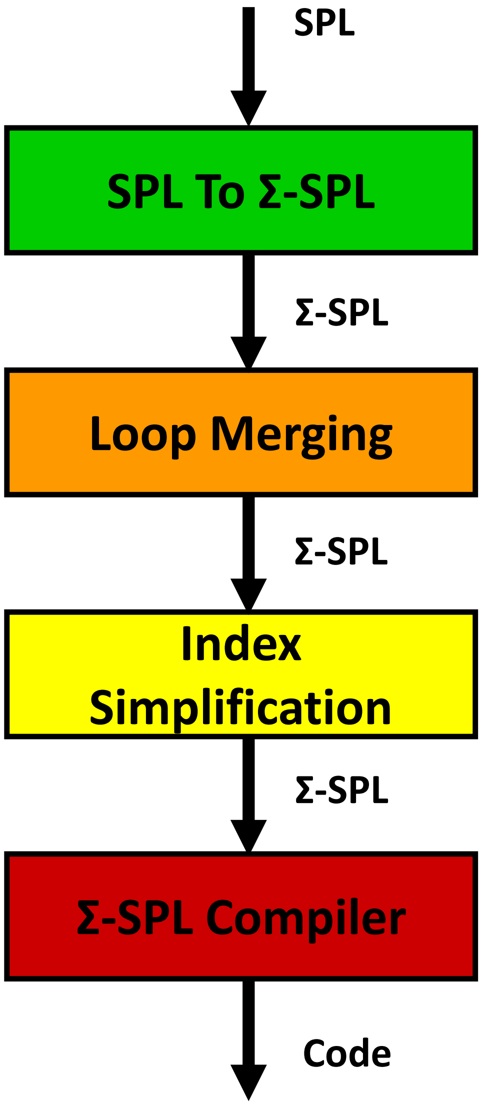
- **Every Σ-SPL formula still represents a matrix factorization**

**Example:** $(I_4 \otimes F_2) \rightarrow \sum_{j=0}^{3} S_{f_j} F_2 G_{f_j}$

$$\begin{bmatrix} F_2 & & & \\ & F_2 & & \\ & & F_2 & \\ & & & F_2 \end{bmatrix}$$



# Loop Merging With Rewriting Rules



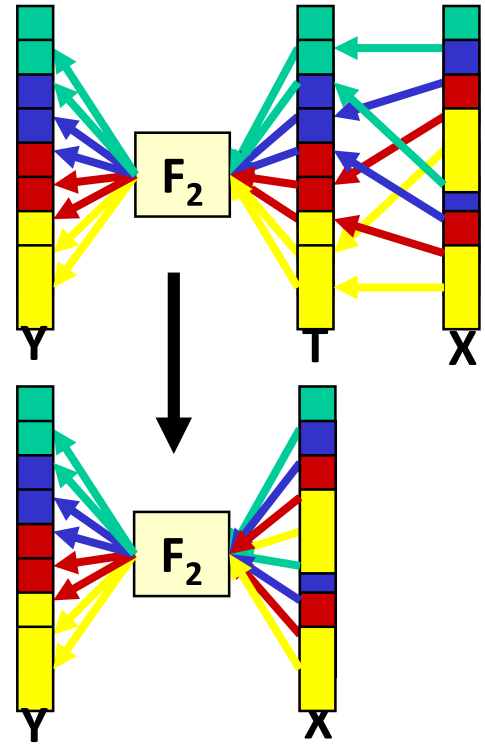

$$y = (\mathrm{I}_4 \otimes \mathrm{F}_2)\, \mathrm{L}^8_4\, x$$

$$\left(\sum_{j=0}^{3} \mathrm{S}_{(j)_4\otimes \imath_2}\, \mathrm{F}_2\, \mathrm{G}_{(j)_4\otimes\imath_2}\right) \mathrm{Perm}_{\ell^8_4}$$

$$\left(\sum_{j=0}^{3} \mathrm{S}_{(j)_4\otimes \imath_2}\, \mathrm{F}_2\, \mathrm{G}_{\ell^8_4\circ((j)_4\otimes\imath_2)}\right)$$

$$\left(\sum_{j=0}^{3} \mathrm{S}_{(j)_4\otimes \imath_2}\, \mathrm{F}_2\, \mathrm{G}_{\imath_2\otimes(j)_4}\right)$$

```
for (int j=0; j<=3; j++) {
        y[2*j]   = x[j] + x[j+4];
        y[2*j+1] = x[j] - x[j+4];
}
```




**Rules:**

$$\mathrm{G}_r\, \mathrm{Perm}_p \rightarrow \mathrm{G}_{p\circ r}$$

$$\ell^{mn}_m \circ \left((j)_m \otimes \imath_n\right) \rightarrow \imath_n \otimes (j)_m$$

# Application: Loop Merging For FFTs

**DFT breakdown rules:**

**Cooley-Tukey FFT** $\mathbf{DFT}_{km} \rightarrow (\mathbf{DFT}_k \otimes \mathrm{I}_m)\, \mathsf{T}^{km}_m (\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathsf{L}^{km}_k$

**Prime factor FFT** $\mathbf{DFT}_{km} \rightarrow \mathsf{V}^T_{k,m} (\mathbf{DFT}_k \otimes \mathrm{I}_m)(\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathsf{V}_{k,m}$

$\mathsf{gcd}(k, m) = 1$

**Rader FFT** $\mathbf{DFT}_p \rightarrow \mathsf{W}^T_p (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) D_p (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1})\, \mathsf{W}_p$

$p$ - prime

**Index mapping functions are non-trivial:**

$$\mathsf{L}^{km}_k \rightarrow \mathsf{Perm}_{\ell^{km}_k} \qquad \ell^{km}_k(i) = \left\lfloor \frac{i}{m} \right\rfloor + k(i \bmod m)$$

$$\mathsf{V}_{k,m} \rightarrow \mathsf{Perm}_{v_{k,m}} \qquad v_{k,m}(i) = \left( m \left\lfloor \frac{i}{m} \right\rfloor + k(i \bmod m) \right) \bmod km$$

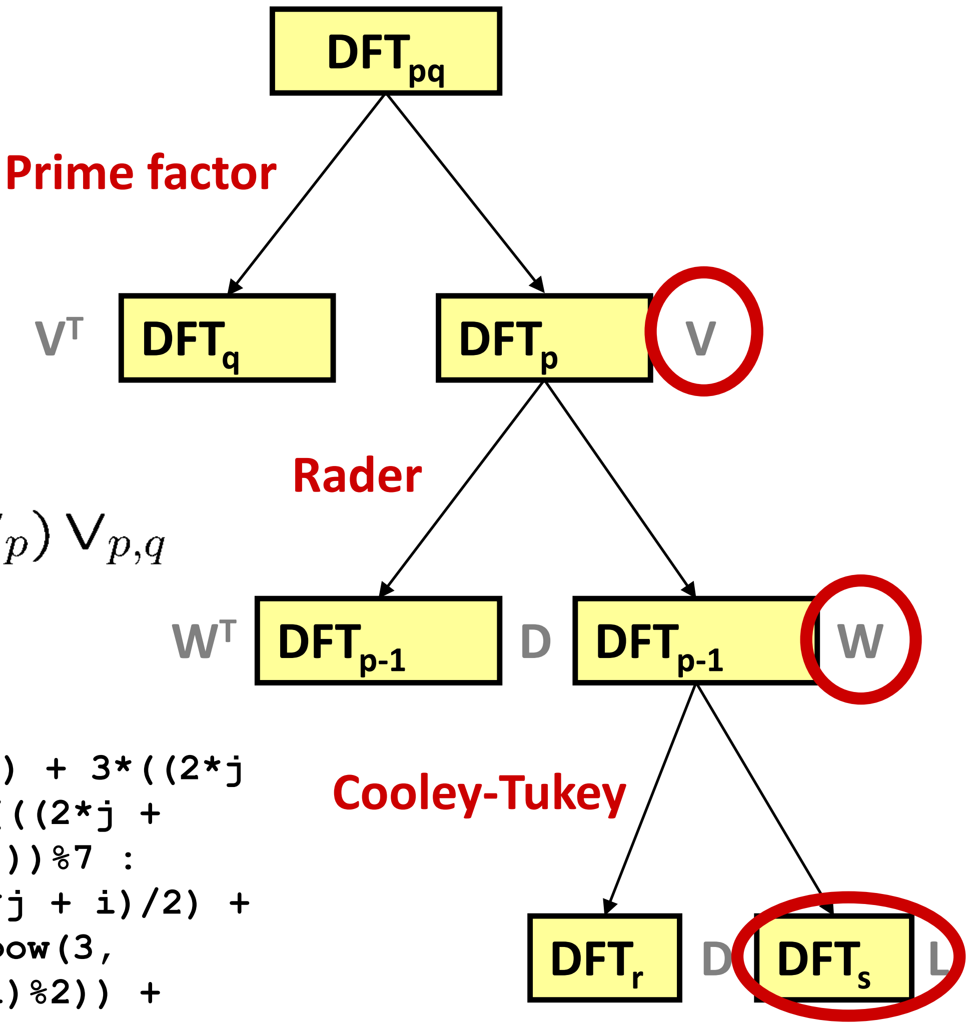
$$\mathsf{W}_p \rightarrow \mathsf{Perm}_{w^p_{1,g}} \qquad w^p_{\phi,g}(i) = \begin{cases} 0, & i = 0, \\ \phi g^i \bmod p, & \text{else.} \end{cases}$$

# Example

**Given $DFT_{pq}$**

p – prime

p-1 = rs




**Formula fragment**

$$(\mathrm{I}_p \otimes (\mathrm{I}_1 \oplus (\mathrm{I}_r \otimes \mathrm{DFT}_s)\, \mathrm{L}^{rs}_r)\, \mathrm{W}_p)\, \mathrm{V}_{p,q}$$

**Code for one memory access**

```
p=7; q=4; r=3; s=2;
t=x[((21*((7*k + ((((((2*j + i)/2) + 3*((2*j
     + i)%2)) + 1)) ? (5*pow(3, ((((2*j +
     i)/2) + 3*((2*j + i)%2)) + 1)))%7 :
     (0)))/7) + 8*((7*k + ((((((2*j + i)/2) +
     3*((2*j + i)%2)) + 1)) ? (5*pow(3,
     ((((2*j + i)/2) + 3*((2*j + i)%2)) +
     1)))%7 : (0)))%7))%28)];
```

**Task:** Index simplification

# Index Simplification: Basic Idea

**Example:** Identity necessary for fusing successive

Rader and prime-factor step

$$\left(\varphi g^{(b+si) \bmod N'}\right) \bmod N = \left((\varphi g^b)(g^s)^i\right) \bmod N$$

$$s|N', \; N'|N, \; 0 \le i < n$$

Performed at the Σ-SPL level through rewrite rules on function objects:

$$\overline{w}_{\phi,g}^{N' \rightarrow N} \circ \overline{h}_{b,s}^{n \rightarrow N'} \quad \longrightarrow \quad \overline{w}_{\phi g^b, g^s}^{n \rightarrow N}$$

**Advantages:**

no analysis necessary

efficient (or doable at all)

```
// Input: _Complex double x[28], output: y[28]
double t1[28];
for(int i5 = 0; i5 <= 27; i5++)
    t1[i5] = x[(7*3*(i5/7) + 4*2*(i5%7))%28];
for(int i1 = 0; i1 <= 3; i1++) {
    double t3[7], t4[7], t5[7];
    for(int i6 = 0; i6 <= 6; i6++)
        t5[i6] = t1[7*i1 + i6];
    for(int i8 = 0; i8 <= 6; i8++)
        t4[i8] = t5[i8 ? (5*pow(3, i8))%7 : 0];
    {
        double t7[1], t8[1];
        t8[0] = t4[0];
        t7[0] = t8[0];
        t3[0] = t7[0];
    }
    {
        double t10[6], t11[6], t12[6];
        for(int i13 = 0; i13 <= 5; i13++)
            t12[i13] = t4[i13 + 1];
        for(int i14 = 0; i14 <= 5; i14++)
            t11[i14] = t12[(i14/2) + 3*(i14%2)];
        for(int i3 = 0; i3 <= 2; i3++) {
            double t14[2], t15[2];
            for(int i15 = 0; i15 <= 1; i15++)
                t15[i15] = t11[2*i3 + i15];
            t14[0] = (t15[0] + t15[1]);
            t14[1] = (t15[0] - t15[1]);
            for(int i17 = 0; i17 <= 1; i17++)
                 t10[2*i3 + i17] = t14[i17];
        }
        for(int i19 = 0; i19 <= 5; i19++)
            t3[i19 + 1] = t10[i19];
    }
    for(int i20 = 0; i20 <= 6; i20++)
        y[7*i1 + i20] = t3[i20];
}
```

```
// Input: _Complex double x[28], output: y[28]
  int p1, b1;
  for(int j1 = 0; j1 <= 3; j1++) {
     y[7*j1] = x[(7*j1%28)];
     p1 = 1; b1 = 7*j1;
     for(int j0 = 0; j0 <= 2; j0++) {
        y[b1 + 2*j0 + 1] = x[(b1 + 4*p1)%28] +
                           x[(b1 + 24*p1)%28];
        y[b1 + 2*j0 + 2] = x[(b1 + 4*p1)%28] -
                           x[(b1 + 24*p1)%28];
        p1 = (p1*3%7);
     }
  }
```

**After, 2 Loops**

**Before, 11 Loops**

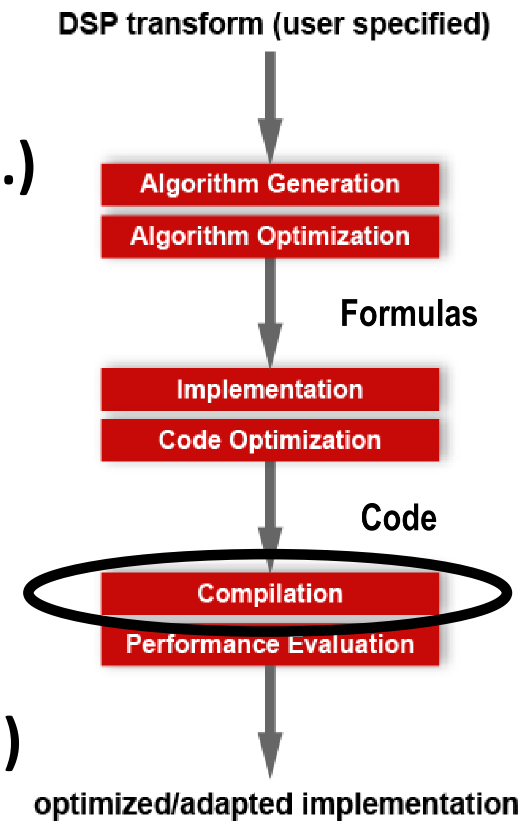
# Organization

- SPL: Problem and algorithm specification
- Σ-SPL: Automating high level optimization
- **Rewriting: Formal parallelization**
- Rewriting: Vectorization
- Verification
- Spiral as FFTX backend
- Summary

F. Franchetti, Y. Voronenko, S. Chellappa, J. M. F. Moura, and M. Püschel
**Discrete Fourier Transform on Multicores: Algorithms and Automatic Implementation**
IEEE Signal Processing Magazine, special issue on “Signal Processing on Platforms with Multiple Cores”, 2009.

# One Approach for All Types of Parallelism

- Shared Memory (Multicore)
- Vector SIMD (SSE, VMX, Double FPU…)
- Message Passing (Clusters)
- Graphics Processors (GPUs)
- FPGA
- HW/SW partitioning
- Multiple Levels of Parallelism (Cell BE)

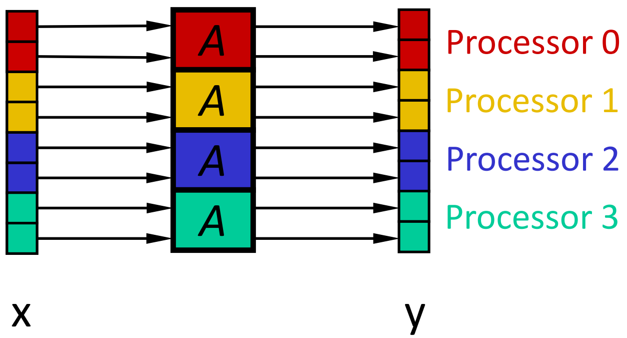



*Spiral: One methodology optimizes for all types of parallelism*

# SPL to Shared Memory Code: Basic Idea

- **Good construct: tensor product**

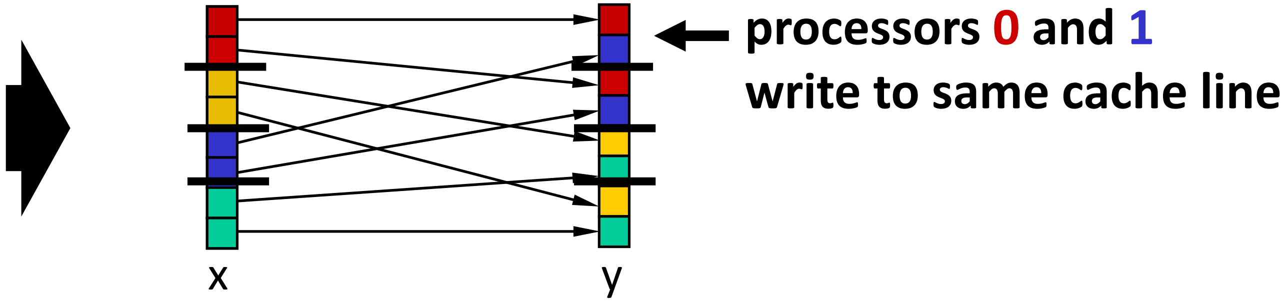
$$y = \left(\mathrm{I}_p \otimes A\right) x$$




**p-way embarrassingly parallel, load-balanced**

- **Problematic construct: permutations produce false sharing**

$$y = \mathrm{L}^8_4 \, x$$




***Task: Rewrite formulas to***
***extract tensor product + treat cache lines as atomic***

# Step 1: Shared Memory Tags

- **Identify crucial hardware parameters**
  - Number of processors: $p$
  - Cache line size: $\mu$

- **Introduce them as tags in SPL:**

$$\underbrace{A}_{\mathrm{smp}(p,\mu)}$$

**This means: formula *A* is to be optimized for *p* processors and cache line size *μ***

- **Tags express hardware constraints within the rewriting system**

# Step 2: Identify "Good" Formulas

- **Load balanced, avoiding false sharing**

$$y = \left(\mathrm{I}_p \otimes A\right) x \quad \text{with} \quad A \in \mathbb{C}^{m\mu \times m\mu}$$

$$y = \left(\bigoplus_{i=0}^{p-1} A_i\right) x \quad \text{with} \quad A_i \in \mathbb{C}^{m\mu \times m\mu}$$

$$y = \left(P \otimes \mathrm{I}_\mu\right) x \quad \text{with } P \text{ a permutation matrix}$$

- **Tagged operators (no further rewriting necessary)**

$$\mathrm{I}_p \otimes_{\|} A, \quad \bigoplus_{i=0}^{p-1}{}_{\|} A_i, \quad P \bar{\otimes} \mathrm{I}_\mu$$

- **Definition: A formula is fully optimized if it is one of the above or of the form**

$$\mathrm{I}_m \otimes A \quad \text{or} \quad AB$$

**where A and B are fully optimized.**

# Step 3: Identify Rewriting Rules

- **Goal: Transform formulas into fully optimized formulas**
  - Formulas rewritten, tags propagated
  - There may be choices

$$\underbrace{AB}_{\mathsf{smp}(p,\mu)} \rightarrow \underbrace{A}_{\mathsf{smp}(p,\mu)} \underbrace{B}_{\mathsf{smp}(p,\mu)}$$

$$\underbrace{A_m \otimes \mathrm{I}_n}_{\mathsf{smp}(p,\mu)} \rightarrow \underbrace{\left(\mathsf{L}_m^{mp} \otimes \mathrm{I}_{n/p}\right)\left(\mathrm{I}_p \otimes (A_m \otimes \mathrm{I}_{n/p})\right)\left(\mathsf{L}_p^{mp} \otimes \mathrm{I}_{n/p}\right)}_{\mathsf{smp}(p,\mu)}$$

$$\underbrace{\mathsf{L}_m^{mn}}_{\mathsf{smp}(p,\mu)} \rightarrow \begin{cases} \underbrace{\left(\mathrm{I}_p \otimes \mathsf{L}_{m/p}^{mn/p}\right)}_{\mathsf{smp}(p,\mu)} \underbrace{\left(\mathsf{L}_p^{pn} \otimes \mathrm{I}_{m/p}\right)}_{\mathsf{smp}(p,\mu)} \\ \underbrace{\left(\mathsf{L}_m^{pm} \otimes \mathrm{I}_{n/p}\right)}_{\mathsf{smp}(p,\mu)} \underbrace{\left(\mathrm{I}_p \otimes \mathsf{L}_m^{mn/p}\right)}_{\mathsf{smp}(p,\mu)} \end{cases}$$

$$\underbrace{\mathrm{I}_m \otimes A_n}_{\mathsf{smp}(p,\mu)} \rightarrow \mathrm{I}_p \otimes_{\|} \left(\mathrm{I}_{m/p} \otimes A_n\right)$$

$$\underbrace{(P \otimes \mathrm{I}_n)}_{\mathsf{smp}(p,\mu)} \rightarrow \left(P \otimes \mathrm{I}_{n/\mu}\right) \overline{\otimes}\, \mathrm{I}_\mu$$

# Simple Rewriting Example

$$\underbrace{A_m \otimes \mathrm{I}_n}_{\mathsf{smp}(p,\mu)}$$

**Loop tiling and scheduling**
**hw-conscious (knows *p* and μ)**

$$\left(\mathsf{L}^{mp}_{m} \otimes \mathrm{I}_{n/p}\right)\left(\mathrm{I}_p \otimes_{\|} (A_m \otimes \mathrm{I}_{n/p})\right)\left(\mathsf{L}^{mp}_{p} \otimes \mathrm{I}_{n/p}\right)$$

**fully optimized**

```
parallel for (i=0; i<p; i++)
   for (j=0; j<n/p; j++)
      y[i*n/p+j:n:i*n/p+j+m-1] =
         A(x[i*n/p+j:n:i*n/p+j+m-1]);
```

# Parallelization by Rewriting

$$\underbrace{\mathrm{DFT}_{mn}}_{\mathsf{smp}(p,\mu)} \;\rightarrow\; \underbrace{\Big((\mathrm{DFT}_m \otimes \mathrm{I}_n)\mathrm{T}^{mn}_n(\mathrm{I}_m \otimes \mathrm{DFT}_n)\,\mathrm{L}^{mn}_m\Big)}_{\mathsf{smp}(p,\mu)}$$

$$\ldots$$

$$\rightarrow\; \underbrace{\Big(\mathrm{DFT}_m \otimes \mathrm{I}_n\Big)}_{\mathsf{smp}(p,\mu)}\; \underbrace{\mathrm{T}^{mn}_n}_{\mathsf{smp}(p,\mu)}\; \underbrace{\Big(\mathrm{I}_m \otimes \mathrm{DFT}_n\Big)}_{\mathsf{smp}(p,\mu)}\; \underbrace{\mathrm{L}^{nm}_m}_{\mathsf{smp}(p,\mu)}$$

$$\ldots$$

$$\rightarrow\; \underline{\Big((\mathrm{L}^{mp}_m \otimes \mathrm{I}_{n/p\mu})\bar{\otimes}\,\mathrm{I}_\mu\Big)}\,\underline{\Big(\mathrm{I}_p \otimes_{\|} (\mathrm{DFT}_m \otimes \mathrm{I}_{n/p})\Big)}\,\underline{\Big((\mathrm{L}^{mp}_p \otimes \mathrm{I}_{n/p\mu})\bar{\otimes}\,\mathrm{I}_\mu\Big)}$$

$$\underline{\left(\bigoplus_{i=0}^{p-1}{}_{\|}\;\mathrm{T}^{mn,i}_n\right)}\;\underline{\Big(\mathrm{I}_p \otimes_{\|} (\mathrm{I}_{m/p} \otimes \mathrm{DFT}_n)\Big)}\,\underline{\Big(\mathrm{I}_p \otimes_{\|} \mathrm{L}^{mn/p}_{m/p}\Big)}\,\underline{\Big((\mathrm{L}^{pn}_p \otimes \mathrm{I}_{m/p\mu})\bar{\otimes}\,\mathrm{I}_\mu\Big)}$$

**Fully optimized (load-balanced, no false sharing)**
**in the sense of our definition**

# Same Approach for Other Parallel Paradigms

## Message Passing:

$$\underbrace{\mathbf{DFT}_{mn}}_{\mathsf{msg}(p,\mu)} \rightarrow \underbrace{\Big((\mathbf{DFT}_m \otimes \mathrm{I}_n)\mathsf{T}^{mn}_n(\mathrm{I}_m \otimes \mathbf{DFT}_n)\,\mathsf{L}^{mn}_m\Big)}_{\mathsf{msg}(p,\mu)}$$

$$\ldots$$

$$\rightarrow \underbrace{\Big(\mathbf{DFT}_m \otimes \mathrm{I}_n\Big)}_{\mathsf{msg}(p,\mu)} \underbrace{\mathsf{T}^{mn}_n}_{\mathsf{msg}(p,\mu)} \underbrace{\Big(\mathrm{I}_m \otimes \mathbf{DFT}_n\Big)}_{\mathsf{msg}(p,\mu)} \underbrace{\mathsf{L}^{nm}_m}_{\mathsf{msg}(p,\mu)}$$

$$\ldots$$

$$\rightarrow \Big((\mathsf{L}^{mp}_m \otimes \mathrm{I}_{n/p\mu})\bar{\otimes}\, \mathrm{I}_\mu\Big)\Big(\mathrm{I}_p \otimes_{\|} (\mathbf{DFT}_m \otimes \mathrm{I}_{n/p})\Big)\Big((\mathsf{L}^{mp}_p \otimes \mathrm{I}_{n/p\mu})\bar{\otimes}\, \mathrm{I}_\mu\Big)$$
$$\left(\bigoplus_{i=0}^{p-1}{}_{\|}\, \mathsf{T}^{mn,i}_n\right)\Big(\mathrm{I}_p \otimes_{\|} (\mathrm{I}_{m/p} \otimes \mathbf{DFT}_n)\Big)\Big(\mathrm{I}_p \otimes_{\|} \mathsf{L}^{mn/p}_{m/p}\Big)\Big((\mathsf{L}^{pn}_p \otimes \mathrm{I}_{m/p\mu})\bar{\otimes}\, \mathrm{I}_\mu\Big)$$

## Vectorization:

$$\underbrace{\Big(\overline{\mathbf{DFT}_{mn}}\Big)}_{\mathsf{vec}(\nu)} \rightarrow \underbrace{\overline{\Big((\mathbf{DFT}_m \otimes \mathrm{I}_n)\mathsf{T}^{mn}_n(\mathrm{I}_m \otimes \mathbf{DFT}_n)\,\mathsf{L}^{mn}_m\Big)}}_{\mathsf{vec}(\nu)}$$

$$\ldots$$

$$\rightarrow \overleftarrow{(\underbrace{\mathsf{DFT}_m \otimes \mathrm{I}_n}_{\mathsf{vec}(\nu)})}^{\nu}\ \overline{(\underbrace{\mathsf{T}^{mn}_n}_{\mathsf{vec}(\nu)})}^{\nu}\ \overrightarrow{\underbrace{(\mathrm{I}_m \otimes \mathsf{DFT}_n)\,\mathsf{L}^{mn}_m}_{\mathsf{vec}(\nu)}}^{\nu}$$

$$\ldots$$

$$\rightarrow (\mathrm{I}_{mn/\nu} \otimes \underbrace{\mathsf{L}^{2\nu}_\nu}_{\mathsf{sse}})(\overline{\mathsf{DFT}_m \otimes \mathrm{I}_{n/\nu}}\vec{\otimes}\, \mathrm{I}_\nu)\overline{(\underbrace{\mathsf{T}^{mn}_n}_{\mathsf{sse}})}^{\nu}$$
$$\Big(\mathrm{I}_{m/\nu} \otimes (\overline{\mathsf{L}^n_\nu}\vec{\otimes}\, \mathrm{I}_\nu)(\mathrm{I}_{n/\nu} \otimes (\mathsf{L}^{2\nu}_\nu \vec{\otimes}\, \mathrm{I}_\nu)(\mathrm{I}_2 \otimes \underbrace{\mathsf{L}^{\nu^2}_\nu}_{\mathsf{sse}})(\mathsf{L}^{2\nu}_2 \vec{\otimes}\, \mathrm{I}_\nu))(\overline{\mathsf{DFT}_n}\vec{\otimes}\, \mathrm{I}_\nu)\Big)$$
$$\Big((\mathsf{L}^{mn}_m \otimes \mathrm{I}_2)\vec{\otimes}\, \mathrm{I}_\nu\Big)(\mathrm{I}_{mn/\nu} \otimes \underbrace{\mathsf{L}^{2\nu}_2}_{\mathsf{sse}})$$

## GPUs:

$$\underbrace{\Big(\mathbf{DFT}_{r^k}\Big)}_{\mathsf{gpu}(t,c)} \rightarrow \underbrace{\left(\prod_{i=0}^{k-1} \mathsf{L}^{r^k}_r \Big(\mathrm{I}_{r^{k-1}} \otimes \mathbf{DFT}_r\Big)\Big(\mathsf{L}^{r^k}_{r^{k-i-1}}(\mathrm{I}_{r^i} \otimes \mathsf{T}^{r^{k-i}}_{r^{k-i-1}})\underbrace{\mathsf{L}^{r^k}_{r^{i+1}}}_{\mathsf{vec}(c)}\Big)\right)\mathsf{R}^{r^k}_r}_{\mathsf{gpu}(t,c)}$$

$$\ldots$$

$$\rightarrow \left(\prod_{i=0}^{k-1} (\mathsf{L}^{r^n/2}_r \vec{\otimes}\, \mathrm{I}_2)\Big(\mathrm{I}_{r^{n-1}/2} \otimes_\times \underbrace{(\mathbf{DFT}_r \vec{\otimes}\, \mathrm{I}_2)\,\mathsf{L}^{2r}_r}_{\mathsf{shd}(t,c)}\Big)\mathsf{T}_i\right)$$
$$(\mathsf{L}^{r^n/2}_r \vec{\otimes}\, \mathrm{I}_2)(\mathrm{I}_{r^{n-1}/2} \otimes_\times \underbrace{\mathsf{L}^{2r}_r}_{\mathsf{shd}(t,c)})(\mathsf{R}^{r^{n-1}}_r \vec{\otimes}\, \mathrm{I}_r)$$

## Verilog for FPGAs:

$$\underbrace{\Big(\mathbf{DFT}_{r^k}\Big)}_{\mathsf{stream}(r^s)} \rightarrow \underbrace{\left[\prod_{i=0}^{k-1} \mathsf{L}^{r^k}_r \Big(\mathrm{I}_{r^{k-1}} \otimes \mathbf{DFT}_r\Big)\Big(\mathsf{L}^{r^k}_{r^{k-i-1}}(\mathrm{I}_{r^i} \otimes \mathsf{T}^{r^{k-i}}_{r^{k-i-1}})\,\mathsf{L}^{r^k}_{r^{i+1}}\Big)\right]\mathsf{R}^{r^k}_r}_{\mathsf{stream}(r^s)}$$

$$\ldots$$

$$\rightarrow \left[\prod_{i=0}^{k-1} \underbrace{\mathsf{L}^{r^k}_r}_{\mathsf{stream}(r^s)} \underbrace{\Big(\mathrm{I}_{r^{k-1}} \otimes \mathbf{DFT}_r\Big)}_{\mathsf{stream}(r^s)} \underbrace{\Big(\mathsf{L}^{r^k}_{r^{k-i-1}}(\mathrm{I}_{r^i} \otimes \mathsf{T}^{r^{k-i}}_{r^{k-i-1}})\,\mathsf{L}^{r^k}_{r^{i+1}}\Big)}_{\mathsf{stream}(r^s)}\right] \underbrace{\mathsf{R}^{r^k}_r}_{\mathsf{stream}(r^s)}$$

$$\ldots$$

$$\rightarrow \left[\prod_{i=0}^{k-1} \underbrace{\mathsf{L}^{r^k}_r}_{\mathsf{stream}(r^s)} \Big(\mathrm{I}_{r^{k-s-1}} \otimes_s (\mathrm{I}_{r^{s-1}} \otimes \mathbf{DFT}_r)\Big) \underbrace{\mathsf{T}'_i}_{\mathsf{stream}(r^s)}\right] \underbrace{\mathsf{R}^{r^k}_r}_{\mathsf{stream}(r^s)}$$

- **Rigorous, correct by construction**
- **Overcomes compiler limitations**

# Autotuning in Constraint Solution Space

$\underbrace{\mathrm{DFT}_8}_{\mathrm{AVX(2\text{-}way}\ \mathbb{C})}$

AVX 2-way
`_Complex double`

DFT$_8$

**Base cases**

$A^{n\times n}\vec{\otimes}\mathrm{I}_2$

$\underbrace{\mathrm{L}^4_2}_{\mathrm{vec}(2)}$

$\underbrace{\mathrm{T}^{mn}_n}_{\mathrm{vec}(2)}$

**Transformation rules**

$(\mathrm{I}_m \otimes A^{n\times n})\mathrm{L}^{mn}_m \rightarrow \left(\mathrm{I}_{m/\nu} \otimes \mathrm{L}^{n\nu}_\nu (A^{n\times n}\otimes \mathrm{I}_\nu)\right)(\mathrm{L}^{mn/\nu}_{m/\nu}\otimes \mathrm{I}_\nu)$

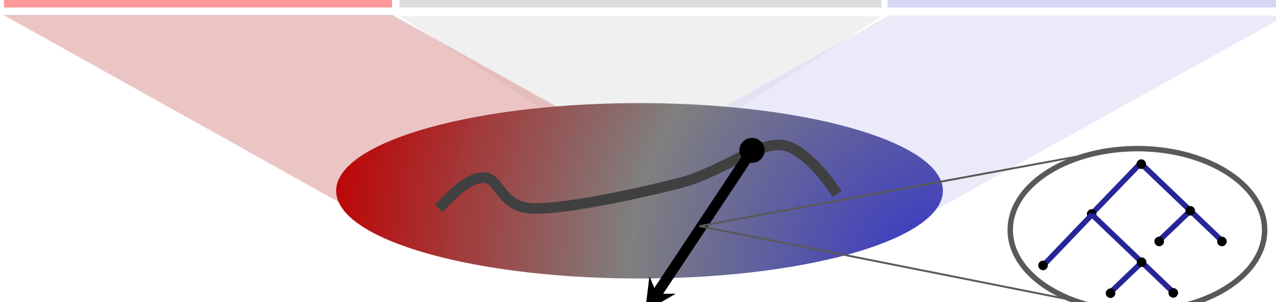
$\mathrm{L}^{n\nu}_\nu \rightarrow (\mathrm{L}^n_\nu \otimes \mathrm{I}_\nu)(\mathrm{I}_{n/\nu}\otimes \mathrm{L}^{\nu^2}_\nu)$

$A^{m\times m}\otimes \mathrm{I}_n \rightarrow (A^{m\times m}\otimes \mathrm{I}_{n/\nu})\otimes \mathrm{I}_\nu$

**Breakdown rules**

$\mathrm{DFT}_{mn} \rightarrow (\mathrm{DFT}_m \otimes \mathrm{I}_n)\mathrm{T}^{mn}_n (\mathrm{I}_m \otimes \mathrm{DFT}_n)\mathrm{L}^{mn}_m$

$\mathrm{DFT}_2 \rightarrow \mathrm{F}_2$



$\left((\mathrm{F}_2\otimes \mathrm{I}_2)\mathrm{T}^4_2(\mathrm{I}_2\otimes \mathrm{F}_2)\mathrm{L}^4_2\vec{\otimes}\mathrm{I}_2\right)\underbrace{\mathrm{T}^8_2}_{\mathrm{vec}(2)}\left(\mathrm{I}_2\otimes \underbrace{\mathrm{L}^4_2}_{\mathrm{vec}(2)}(\mathrm{F}_2\vec{\otimes}\mathrm{I}_2)\right)\left(\mathrm{L}^4_2\vec{\otimes}\mathrm{I}_2\right)$

SPL specification
↓ Expansion + backtracking
SPL (dataflow) expression
↓ Recursive descent
Σ-SPL (loop) expression
↓ Confluent term rewriting
Optimized Σ-SPL expression
↓ Recursive descent
Abstract code
↓ Confluent term rewriting
Optimized abstract code
↓ Recursive descent
C code

# Translating an OL Expression Into Code

**Constraint Solver Input:** $\underbrace{\mathrm{DFT}_8}_{\mathrm{AVX(2\text{-}way}\ \mathbb{C})}$

**Output =**

**Ruletree, expanded into**

See Figure 5

## SPL Expression:

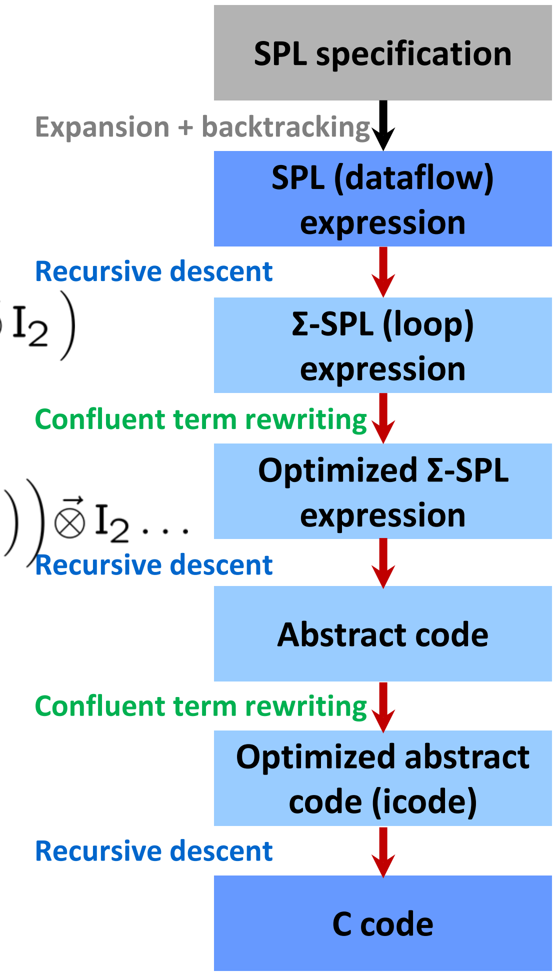
$$\Big((\mathrm{F}_2 \otimes \mathrm{I}_2)\mathrm{T}^4_2(\mathrm{I}_2 \otimes \mathrm{F}_2)\mathrm{L}^4_2 \vec{\otimes}\, \mathrm{I}_2\Big)\ \underbrace{\mathrm{T}^8_2}_{\mathrm{vec}(2)}\ \Big(\mathrm{I}_2 \otimes \underbrace{\mathrm{L}^4_2}_{\mathrm{vec}(2)}\ (\mathrm{F}_2 \vec{\otimes}\, \mathrm{I}_2)\Big)\Big(\mathrm{L}^4_2 \vec{\otimes}\, \mathrm{I}_2\Big)$$

## Σ-SPL:

$$\Big(\sum_{j=0}^{1}\big(\mathrm{S}_{\imath_2 \otimes (j)_2}\mathrm{F}_2\mathrm{Diag}^2_{x \mapsto \omega_4^{2i+j}x}\mathrm{G}_{\imath_2 \otimes (j)_2}\big)\sum_{j=0}^{1}\big(\mathrm{S}_{(j)_2 \otimes \imath_2}\mathrm{F}_2\mathrm{G}_{\imath_2 \otimes (j)_2}\big)\Big)\vec{\otimes}\, \mathrm{I}_2 \ldots$$

## C Code:

```
void dft8(_Complex double *Y, _Complex double *X) {
    __m256d s38, s39, s40, s41,...
    __m256d  *a17, *a18;
    a17 = ((__m256d  *) X);
    s38 = *(a17);
    s39 = *((a17 + 2));
    t38 = _mm256_add_pd(s38, s39);
    t39 = _mm256_sub_pd(s38, s39);
    ...
    s52 = _mm256_sub_pd(s45, s50);
    *((a18 + 3)) = s52;
}
```



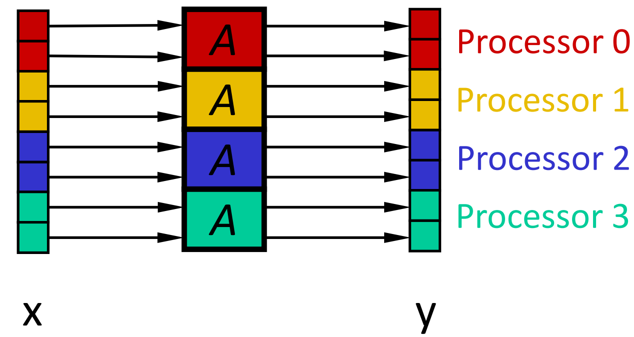
# Discussion

- Parallelization at the mathematical level through rewriting and constraint programming
- Generates a space of "reasonable algorithms" that can be searched for adaptation to memory hierarchy
- Very efficient since no analysis is required
- Principled, domain-specific approach
- Applicable across transforms and parallelism types

# Message Passing

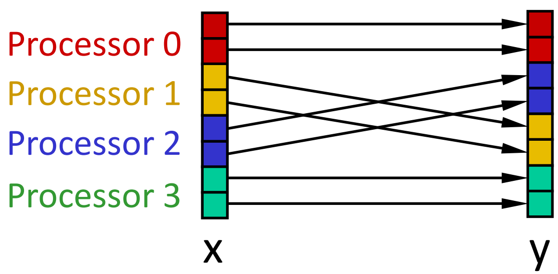
- **Good construct: tensor product**

$$y = \left(\mathrm{I}_p \otimes A\right) x$$




**Characteristics: no communication**

- **Permutations are explicit communication** $y = \left(\mathrm{L}^4_2 \otimes \mathrm{I}_2\right) x$




```
// same program for all processors (SPMD)
switch (processor_num) {
    case 0: copy(y_local, x_local); break;
    case 1: SENDRECV(y_local, x_local, 2); break;
    case 2: SENDRECV(y_local, x_local, 1); break;
    case 3: copy(y_local, x_local); break;
}
```

***Apply same 3-step approach:***

1. ***Identify hw parameters***
2. ***Identify good formulas***
3. ***Identify rewriting rules***

# Parallelization for Distributed Memory

$$\underbrace{\mathrm{DFT}_{mn}}_{\mathrm{par}(p)} \rightarrow \underbrace{(\mathrm{DFT}_m \otimes \mathrm{I}_n)}_{\mathrm{par}(p\leftarrow q)} \underbrace{\mathrm{T}_n^{mn}}_{\mathrm{par}(q)} \underbrace{(\mathrm{I}_m \otimes \mathrm{DFT}_n)}_{\mathrm{par}(q)} \underbrace{\mathrm{L}_m^{mn}}_{\mathrm{par}(q\leftarrow p)}$$

$$\dots$$

$$\dots$$

$$\dots$$

$$\rightarrow \left(\mathrm{I}_p \otimes_{\|} \mathrm{L}_{m/p}^{mn/p}\right) \underbrace{\left(\mathrm{L}_p^{p^2} \otimes \mathrm{I}_{mn/p^2}\right)}_{\mathrm{comm}(p\leftarrow q)} \left(\mathrm{I}_q \otimes_{\|} (\mathrm{I}_{p/q} \otimes \mathrm{L}_p^n \otimes \mathrm{I}_{m/p})\right)\left(\mathrm{I}_q \otimes_{\|} (\mathrm{I}_{n/q} \otimes \mathrm{DFT}_m)\right)$$

$$\left(\mathrm{I}_q \otimes_{\|} \mathrm{L}_{m/q}^{mn/q}\right) \underbrace{\left(\mathrm{L}_q^{q^2} \otimes \mathrm{I}_{mn/q^2}\right)}_{\mathrm{comm}(q)} \left(\mathrm{I}_q \otimes_{\|} (\mathrm{L}_q^n \otimes \mathrm{I}_{m/q})\right) \mathrm{T}_n^{mn} \left(\mathrm{I}_q \otimes_{\|} (\mathrm{I}_{m/q} \otimes \mathrm{DFT}_n)\right)$$

$$\left(\mathrm{I}_q \otimes_{\|} (\mathrm{I}_{p/q} \otimes \mathrm{L}_{m/p}^{mn/p})\right) \underbrace{\left(\mathrm{L}_p^{p^2} \otimes \mathrm{I}_{mn/p^2}\right)}_{\mathrm{comm}(q\leftarrow p)} \left(\mathrm{I}_p \otimes_{\|} (\mathrm{L}_p^n \otimes \mathrm{I}_{m/p})\right)$$

***p*-way parallelized**

**with intermediate computation on *q* processors**

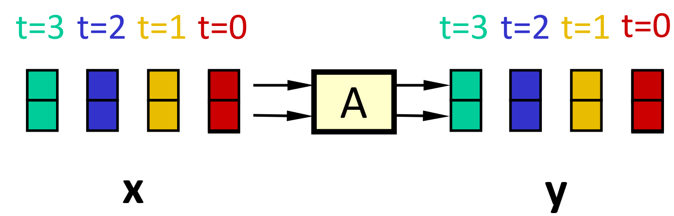
# Parallelization/Streaming for FPGAs

- **Data streams steadily in packets**

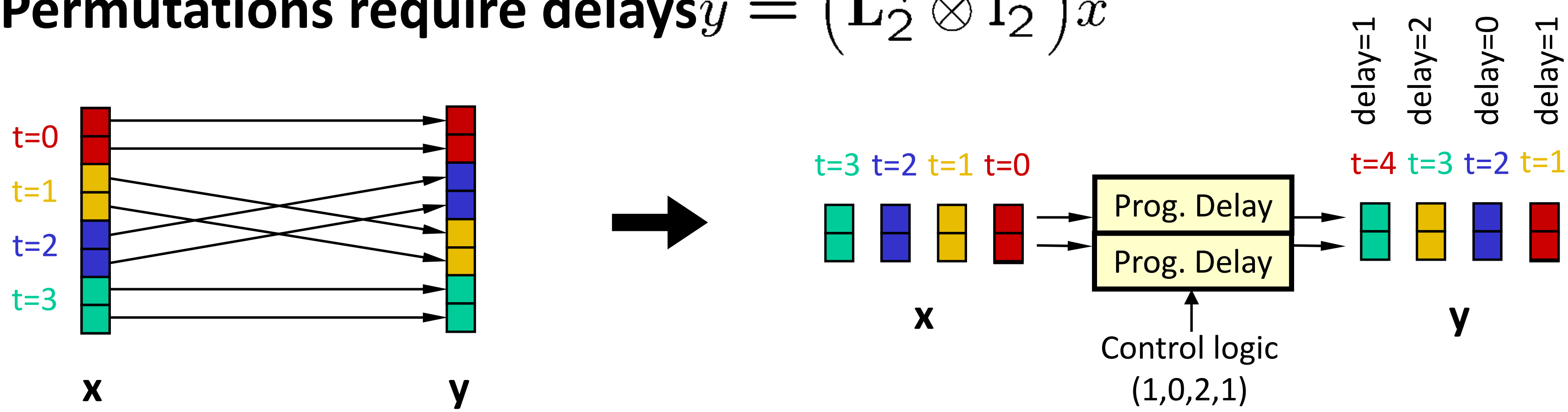
$$y = \left( \mathrm{I}_p \otimes A^{n \times n} \right) x$$

Packet size

Number of packets = time steps




- **Permutations require delays** $y = \left( \mathrm{L}^4_2 \otimes \mathrm{I}_2 \right) x$




*Apply same 3-step approach:*

1. *Identify hw parameters*
2. *Identify good formulas*
3. *Identify rewriting rules*

# Streaming by Rewriting

$$\underbrace{\left(\mathbf{DFT}_{r^k}\right)}_{\mathsf{stream}(r^s)} \rightarrow \underbrace{\left[\prod_{i=0}^{k-1} \mathsf{L}_r^{r^k}\left(\mathrm{I}_{r^{k-1}} \otimes \mathbf{DFT}_r\right)\left(\mathsf{L}_{r^{k-i-1}}^{r^k}(\mathrm{I}_{r^i} \otimes \mathsf{T}_{r^{k-i-1}}^{r^{k-i}})\,\mathsf{L}_{r^{i+1}}^{r^k}\right)\right] \mathsf{R}_r^{r^k}}_{\mathsf{stream}(r^s)}$$

...

$$\rightarrow \left[\prod_{i=0}^{k-1} \underbrace{\mathsf{L}_r^{r^k}}_{\mathsf{stream}(r^s)} \underbrace{\left(\mathrm{I}_{r^{k-1}} \otimes \mathbf{DFT}_r\right)}_{\mathsf{stream}(r^s)} \underbrace{\left(\mathsf{L}_{r^{k-i-1}}^{r^k}(\mathrm{I}_{r^i} \otimes \mathsf{T}_{r^{k-i-1}}^{r^{k-i}})\,\mathsf{L}_{r^{i+1}}^{r^k}\right)}_{\mathsf{stream}(r^s)}\right] \underbrace{\mathsf{R}_r^{r^k}}_{\mathsf{stream}(r^s)}$$

...

$$\rightarrow \left[\prod_{i=0}^{k-1} \underbrace{\mathsf{L}_r^{r^k}}_{\mathsf{stream}(r^s)} \left(\mathrm{I}_{r^{k-s-1}} \otimes_s (\mathrm{I}_{r^{s-1}} \otimes \mathbf{DFT}_r)\right) \underbrace{\mathsf{T}_i'}_{\mathsf{stream}(r^s)}\right] \underbrace{\mathsf{R}_r^{r^k}}_{\mathsf{stream}(r^s)}$$

**streamed at $r^s$ words per cycle**

**base cases**

# Target: Specialized Soft Processor

SPL formula $y = (I_4 \otimes F_2)\, L^8_4 \; x$

C code generation

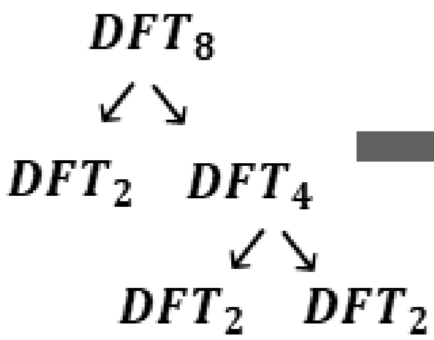
FPGA OpenCL code generation

```
// each call executes full for loop
void sub(double *y, double *x) {
   for (int j=0; j<=3; j++){
      y[2*j] = x[j] + x[j+4];
      y[2*j+1] = x[j] - x[j+4];
   }
}
```

```
// each call executes one iteration of the for loop
__kernel void sub(__global double *y,
    __global double *x, int control) {
   // reinitialize
   if (!control) { j = 0; return; }
   // for (int j=0; j<=3; j++)
   {
      static int j = 0;
      if (j<=3) {
         y[2*j] = x[j] + x[j+4];
         y[2*j+1] = x[j] - x[j+4];
         j++;
      }
   }
}
```

Hardware design issue → SPIRAL algorithm transformation + C coding style issue

# OpenCL State Machine Code Generation




$$\sum_{i3=0}^{1}\left(\sum_{i6=0}^{1} S_{2,i6}A_2G_{2,i6}\cdot\sum_{i7=0}^{1} S_{1,i7}A_1G_{1,i7}\right)\cdot\sum_{i2=0}^{3} S_{0,i2}A_0G_{0,i2}$$

```
for(int i2 = 0; i2 <= 3; i2++) {
    BB(0);
}
for(int i3 = 0; i3 <= 1; i3++) {
    for(int i7 = 0; i7 <= 1; i7++) {
        BB(1);
    }
    for(int i6 = 0; i6 <= 1; i6++) {
        BB(2);
    }
}
```

```
while (true) {
    if (bben0==1 && i2<=3)  {
        // output logic
        output = f(i2);
        // next state logic
        if (i2<3) {
            i2++;
        } else if (i2==3) {
            i2=0; bben0=0;
            bben1=1;
        }
    else if (bben1==1 && i7<=1)  {
        // output logic
        // next state logic
    }
    else if (bben2==1 && i6<=1)  {
        // output logic
        // next state logic
     }
     write_channel(ouput);
}
```

# Organization

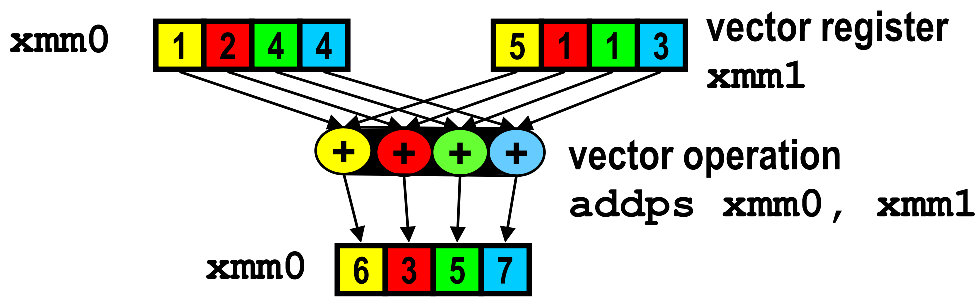
- SPL: Problem and algorithm specification
- Σ-SPL: Automating high level optimization
- Rewriting: Formal parallelization
- **Rewriting: Vectorization**
- Verification
- Spiral as FFTX backend
- Summary

F. Franchetti, M. Püschel
**Short Vector Code Generation for the Discrete Fourier Transform**
Proceedings of the 17th International Parallel and Distributed Processing Symposium (IPDPS '03), pages 58-67.

F. Franchetti and M. Püschel
**Generating SIMD Vectorized Permutations**
Proceedings of International Conference on Compiler Construction (CC) 2008

# SIMD (Signal Instruction Multiple Data) Vector Instructions in a Nutshell

- **What are these instructions?**
  - Extension of the ISA. Data types and instructions for parallel computation on short **(2-way–16-way) vectors** of integers and floats




- Intel MMX
- AMD 3DNow!
- Intel SSE
- AMD Enhanced 3DNow!
- Motorola AltiVec/VMX
- AMD 3DNow! Professional
- Intel SSE2
- IBM BlueGene/L PPC440FP2
- IBM QPX
- IBM VSX
- Intel SSE3
- Intel SSSE3
- Intel SSE4, 4.1, 4.2
- Intel AVX, AVX2
- Intel AVX512

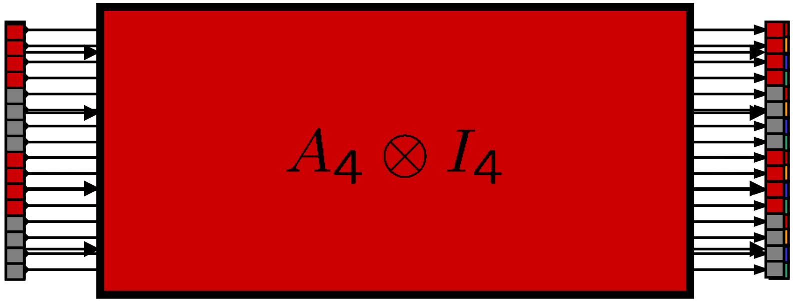
- **Problems:**
  - Not standardized
  - Compiler vectorization limited
  - Low-level issues (data alignment,…)
  - Reordering data kills runtime

***One can easily slow down a program by vectorizing it***

# Vectorization: Basic Idea

- **Good construct: tensor product**

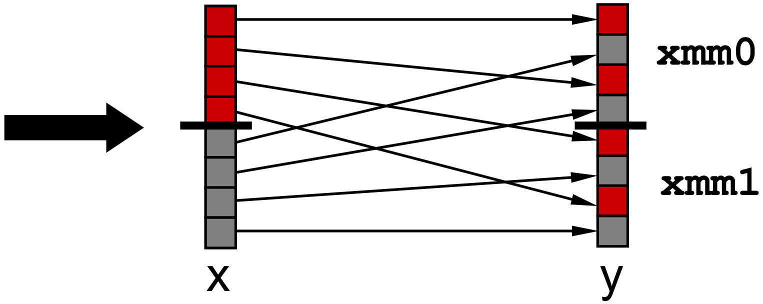
$$y = \left(A \otimes \mathrm{I}_{\nu}\right) x$$




**Characteristics: block operation and alignment preserving**

- **Problematic construct: permutations must be done in register**

$$y = \mathrm{L}^8_4 x$$




***Task: Rewrite formulas to***
***extract tensor product + minimize in-register shuffles***

# SIMD Vectorization: 3-Step Procedure

1. **Identify crucial hardware parameters** $\underbrace{A}_{\mathsf{vec}(\nu)}$
   - **Vector length: ν**

2. **Identify good formulas**
   - **Tensor product: A $I_\nu$**
   - **Base cases:** Build library from shuffle instructions **`unpacklo`**, **`unpackhi`**, **`shufps`**,…
   - **Definition:** Vectorized formula

3. **Identify rewriting rules**

$$\underbrace{\mathrm{I}_n \otimes A^{k\times m}}_{\mathsf{vec}(\nu)} \rightarrow \underbrace{\mathrm{I}_{n/\nu} \otimes \mathrm{L}^{k\nu}_{\nu}}_{\mathsf{vec}(\nu)} \left(A^{k\times m} \vec{\otimes}\, \mathrm{I}_\nu\right) \underbrace{\mathrm{L}^{m\nu}_{m}}_{\mathsf{vec}(\nu)}$$

$$\underbrace{\mathrm{L}^{m\nu}_{m}}_{\mathsf{vec}(\nu)} \rightarrow \left(\mathrm{I}_{m/\nu} \otimes \underbrace{\mathrm{L}^{\nu^2}_{\nu}}_{\mathsf{sse}}\right)\left(\mathrm{L}^{m}_{m/\nu} \vec{\otimes}\, \mathrm{I}_\nu\right)$$

Base case for Intel SSE

# Vectorization by Rewriting

$$\underbrace{\left(\overline{\mathbf{DFT}_{mn}}\right)}_{\mathrm{vec}(\nu)} \rightarrow \underbrace{\left(\overline{(\mathbf{DFT}_m \otimes \mathrm{I}_n)\, \mathrm{T}^{mn}_n\, (\mathrm{I}_m \otimes \mathbf{DFT}_n)\, \mathrm{L}^{mn}_m}\right)}_{\mathrm{vec}(\nu)}$$

$$\ldots$$

$$\rightarrow \underbrace{\left(\overleftarrow{\mathrm{DFT}_m \otimes \mathrm{I}_n}\right)}_{\mathrm{vec}(\nu)}^{\nu} \underbrace{\left(\overline{\mathrm{T}^{mn}_n}\right)}_{\mathrm{vec}(\nu)}^{\nu} \underbrace{\left(\overrightarrow{(\mathrm{I}_m \otimes \mathrm{DFT}_n)\, \mathrm{L}^{mn}_m}\right)}_{\mathrm{vec}(\nu)}^{\nu}$$

$$\ldots$$

$$\rightarrow (\mathrm{I}_{mn/\nu} \otimes \underbrace{\mathrm{L}^{2\nu}_{\nu}}_{\mathrm{sse}})(\overline{\mathrm{DFT}_m \otimes \mathrm{I}_{n/\nu} \vec{\otimes} \mathrm{I}_\nu})\underbrace{(\overline{\mathrm{T}^{mn}_n})}_{\mathrm{sse}}^{\nu}$$

$$\left(\mathrm{I}_{m/\nu} \otimes (\overline{\mathrm{L}^{n}_{\nu}} \vec{\otimes} \mathrm{I}_\nu)(\mathrm{I}_{n/\nu} \otimes (\mathrm{L}^{2\nu}_{\nu} \vec{\otimes} \mathrm{I}_\nu)(\mathrm{I}_2 \otimes \underbrace{\mathrm{L}^{\nu^2}_{\nu}}_{\mathrm{sse}})(\mathrm{L}^{2\nu}_{2} \vec{\otimes} \mathrm{I}_\nu))(\overline{\mathrm{DFT}_n} \vec{\otimes} \mathrm{I}_\nu)\right)$$

$$\left((\mathrm{L}^{mn}_{m} \otimes \mathrm{I}_2) \vec{\otimes} \mathrm{I}_\nu\right)(\mathrm{I}_{mn/\nu} \otimes \underbrace{\mathrm{L}^{2\nu}_{2}}_{\mathrm{sse}})$$

**tensor products**

**base cases**

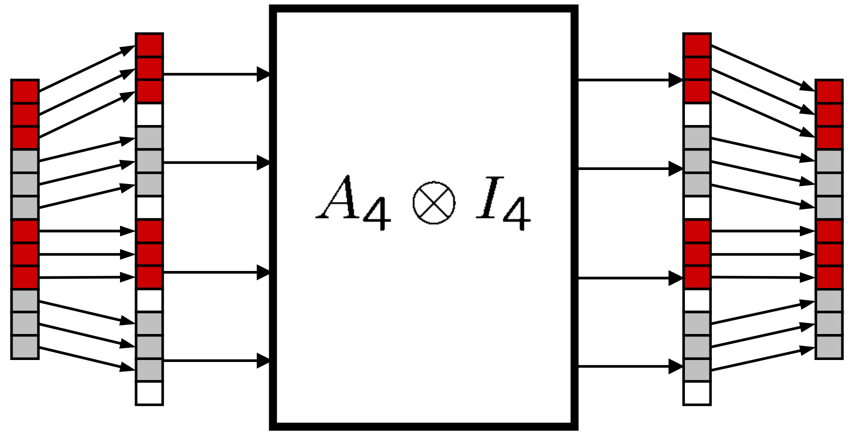
***Formula is vectorized w.r.t. Definition***

# Vectorization of Odd Problem Sizes

**Zero-padding: Load/store *km* elements into/from *m* ν-way vectors**

$$y = (\mathrm{I}_m \otimes \mathrm{I}_{\nu \times k})x$$

$$y = (\mathrm{I}_m \otimes \mathrm{I}_{k \times \nu})x$$




| SPL formula | data type | compiled to |
|---|---|---|
| $\mathrm{I}_{k \times 4}$ | 4-way float | _mm_maskmoveu_si128 + _mm_castps_si128 |
| $\mathrm{I}_{4 \times k}$ | 4-way float | _mm_loadu_ps + _mm_and_i128 |
| $\mathrm{I}_{k \times 8}$ | 8-way short | _mm_maskmoveu_si128 |
| $\mathrm{I}_{8 \times k}$ | 8-way short | _mm_loadu_si128 |

# Automatically Deriving Vector Base Cases


- **Translate SIMD vector ISA into matrix representation**
- **Design rule system to generate *vector matrix formulas***
- **Define cost measure on matrix formulas**
- **Use dynamic programming with backtracking to find vector program with minimal cost**

## Vector matrix formula in BNF

$$\begin{aligned}
\langle \mathsf{vmf} \rangle \; &::= \; \langle \mathsf{vmf} \rangle \, \langle \mathsf{vmf} \rangle \mid \mathrm{I}_m \otimes \langle \mathsf{vmf} \rangle \mid \begin{pmatrix} \langle \mathsf{vmf} \rangle \\ \langle \mathsf{vmf} \rangle \end{pmatrix} \mid \langle \mathsf{perm} \rangle \otimes \mathrm{I}_\nu \mid \\
& \quad \langle \mathsf{perm} \rangle \otimes \mathrm{I}_{\nu/2} \; \text{if } \mathbf{L}_2^4 \otimes \mathrm{I}_{\nu/2} \; \text{possible} \mid M_{\mathtt{instr}} \; \text{with } \mathtt{instr} \text{ in ISA} \\
\langle \mathsf{perm} \rangle \; &::= \; \mathbf{L}_m^{mn} \mid \mathrm{I}_m \otimes \langle \mathsf{perm} \rangle \mid \langle \mathsf{perm} \rangle \otimes \mathrm{I}_m \mid \langle \mathsf{perm} \rangle \langle \mathsf{perm} \rangle
\end{aligned}$$

# Translating Instructions into Matrices

## Intel C++ Compiler Manual

```
__m128 _mm_unpackhi_ps(__m128 a, __m128 b)
r0 := a2; r1 := b2; r2 := a3; r3 := b3
```

## Instruction specification (GAP code)

```
Intel_SSE2.4_x_float._mm_unpackhi_ps := rec(
   v := 4,
   semantics := (a, b, p) -> [a[2], b[2], a[3], b[3]],
   parameters := []
);
```

## SSE instruction as matrix

```
__m128 t, x0, x1;
t = _mm_unpackhi_ps(x0, x1);
```

➡

$$\vec{t} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \end{bmatrix}$$

**Automatically build matrix from `semantics()` function**

# Example: Sequence of Two Instructions

## Instruction set: Intel SSE 4-way float

```
y = _mm_unpacklo_ps(x0, x1);
```

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

```
y = _mm_shuffle_ps(x0, x1,
        _MM_SHUFFLE(1,2,1,2));
```

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

```
y = _mm_shuffle_ps(x0, x1,
        _MM_SHUFFLE(3,4,3,4));
```

$$\begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

## Translating a vector matrix into a instruction sequence

$$L^4_2 \otimes I_2 = \left[\begin{array}{cccccccc} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ \hline 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{array}\right]$$

```
// __m128 *y, *x
Y[0] = _mm_shuffle_ps(x0, x1,
          _MM_SHUFFLE(1,2,1,2));
Y[1] = _mm_shuffle_ps(x0, x1,
          _MM_SHUFFLE(3,4,3,4));
```

# Rule System: Recursive Matrix Factorization

- Recursively factorizes stride permutations
- "Blocking of matrix transposition" in linear memory
- Choices ! Dynamic programming with backtracking
- Trigger ISA-specific termination rules

Start: stride permutation

$$L_m^{mn} \rightarrow I_1 \otimes L_m^{mn} \otimes I_1$$

$$I_\ell \otimes L_n^{kmn} \otimes I_r \rightarrow \left(I_\ell \otimes L_n^{kn} \otimes I_{mr}\right)\left(I_{\ell k} \otimes L_n^{mn} \otimes I_r\right)$$

$$I_\ell \otimes L_n^{kmn} \otimes I_r \rightarrow \left(I_\ell \otimes L_{kn}^{kmn} \otimes I_r\right)\left(I_\ell \otimes L_{mn}^{kmn} \otimes I_r\right)$$

$$I_\ell \otimes L_{km}^{kmn} \otimes I_r \rightarrow \left(I_{k\ell} \otimes L_m^{mn} \otimes I_r\right)\left(I_\ell \otimes L_k^{kn} \otimes I_m\right)$$

$$I_\ell \otimes L_{km}^{kmn} \otimes I_r \rightarrow \left(I_\ell \otimes L_k^{kmn} \otimes I_r\right)\left(I_\ell \otimes L_m^{kmn} \otimes I_r\right)$$

$$I_{k\ell} \otimes L_m^{mn} \otimes I_r \rightarrow I_k \otimes \left(I_\ell \otimes L_m^{mn} \otimes I_r\right) \quad \text{if } \ell mnr \in \{\nu, 2\nu\}$$

Choice: factorize kmn

Triggers termination rules

# Cost Function: Weighted Instruction Count

- **Defines recursive cost function for matrix formulas**
- **Each instruction has an associated cost**
- **Vector assignments are "for free"**

$$
\begin{aligned}
\mathrm{Cost}_{\mathrm{ISA},\nu}(P) &= \infty, \quad P \text{ not a } \langle\mathrm{vmf}\rangle \\
\mathrm{Cost}_{\mathrm{ISA},\nu}(M^{\nu}_{\mathtt{instr}}) &= c_{\mathtt{instr}} \\
\mathrm{Cost}_{\mathrm{ISA},\nu}(P \otimes \mathrm{I}_\nu) &= 0, \quad P \text{ permutation} \\
\mathrm{Cost}_{\mathrm{ISA},\nu}\left(P \otimes \mathrm{I}_{\nu/2}\right) &= \lfloor n/2 \rfloor c_{\mathtt{i1}} + \lceil n/2 \rceil c_{\mathtt{i2}}, \quad P\ 2n \times 2n \text{ permutation} \\
\mathrm{Cost}_{\mathrm{ISA},\nu}(A\,B) &= \mathrm{Cost}_{\mathrm{ISA},\nu}(A) + \mathrm{Cost}_{\mathrm{ISA},\nu}(B) \\
\mathrm{Cost}_{\mathrm{ISA},\nu}\left(\begin{pmatrix} A \\ B \end{pmatrix}\right) &= \mathrm{Cost}_{\mathrm{ISA},\nu}(A) + \mathrm{Cost}_{\mathrm{ISA},\nu}(B) \\
\mathrm{Cost}_{\mathrm{ISA},\nu}(\mathrm{I}_m \otimes A) &= m\,\mathrm{Cost}_{\mathrm{ISA},\nu}(A)
\end{aligned}
$$

# Vector Program: 8-way Vectorized $L_8^{64}$

$$L_8^{64} = \left(I_4 \otimes (L_2^4 \otimes I_4)\right)\left(L_4^8 \otimes I_8\right)\left(I_4 \otimes (L_4^8 \otimes I_2)\right)\left((I_2 \otimes L_2^4) \otimes I_8\right)\left(I_4 \otimes L_8^{16}\right)$$

```
__m128 X[8], Y[8], t3, t4, t7, t8, t11, t12, t15, t16,
       t17, t18, t19, t20, t21, t22, t23, t24;
t3 = _mm_unpacklo_epi16(X[0], X[1]); t4 = _mm_unpackhi_epi16(X[0], X[1]);
t7 = _mm_unpacklo_epi16(X[2], X[3]); t8 = _mm_unpackhi_epi16(X[2], X[3]);
t11 = _mm_unpacklo_epi16(X[4], X[5]);t12 = _mm_unpackhi_epi16(X[4], X[5]);
t15 = _mm_unpacklo_epi16(X[6], X[7]);t16 = _mm_unpackhi_epi16(X[6], X[7]);
t17 = _mm_unpacklo_epi32(t3, t7);     t18 = _mm_unpackhi_epi32(t3, t7);
t19 = _mm_unpacklo_epi32(t4, t8);     t20 = _mm_unpackhi_epi32(t4, t8);
t21 = _mm_unpacklo_epi32(t11, t15);   t22 = _mm_unpackhi_epi32(t11, t15);
t23 = _mm_unpacklo_epi32(t12, t16);   t24 = _mm_unpackhi_epi32(t12, t16);
Y[0] = _mm_unpacklo_epi64(t17, t21); Y[1] = _mm_unpackhi_epi64(t17, t21);
Y[2] = _mm_unpacklo_epi64(t18, t22); Y[3] = _mm_unpackhi_epi64(t18, t22);
Y[4] = _mm_unpacklo_epi64(t19, t23); Y[5] = _mm_unpackhi_epi64(t19, t23);
Y[6] = _mm_unpacklo_epi64(t20, t24); Y[7] = _mm_unpackhi_epi64(t20, t24);
```

***8-way vectorized transposition of 8x8 matrix***

# Organization

- SPL: Problem and algorithm specification
- Σ-SPL: Automating high level optimization
- Rewriting: Formal parallelization
- Rewriting: Vectorization
- **Verification**
- Spiral as FFTX backend
- Summary

# Symbolic Verification

- **Transform = Matrix-vector multiplication**
  **matrix fully defines the operation**

$$\mathrm{DFT}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

**= ?**

- **Algorithm = Formula**
  **represents a matrix expression, can be evaluated to a matrix**

$$(\mathrm{DFT}_2 \otimes \mathrm{I}_2)\,\mathrm{T}^4_2(\mathrm{I}_2 \otimes \mathrm{DFT}_2)\,\mathrm{L}^4_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Empirical Verification

- **Run program on all basis vectors, compare to columns of transform matrix**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

= ?

```
DFT4([0,1,0,0])
```

- **Compare program output on random vectors to output of a random implementation of same kernel**

```
DFT4([0.1,1.77,2.28,-55.3])
```

= ?

```
DFT4_rnd([0.1,1.77,2.28,-55.3]))
```

# Verification of the Generator

- **Rule replaces left-hand side by right-hand side when preconditions match**

$$\mathrm{I}_m \otimes A_n \rightarrow \mathrm{L}^{mn}_m (A_n \otimes \mathrm{I}_m)\, \mathrm{L}^{mn}_n$$

- **Test rule by evaluating expressions before and after rule application and compare result**

$$\mathrm{I}_2 \otimes \mathrm{DFT}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

= ?

$$\mathrm{L}^4_2 (\mathrm{DFT}_2 \otimes \mathrm{I}_2)\, \mathrm{L}^4_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Organization

- SPL: Problem and algorithm specification
- Σ-SPL: Automating high level optimization
- Rewriting: Formal parallelization
- Rewriting: Vectorization
- Verification
- **Spiral as FFTX backend**
- Summary

# Have You Ever Wondered About This?

## Numerical Linear Algebra

**LAPACK**
**ScaLAPACK**
LU factorization
Eigensolves
SVD

**BLAS, BLACS**
BLAS-1
BLAS-2
BLAS-3

## Spectral Algorithms

Convolution
Correlation
Upsampling
Poisson solver
…

?

**FFTW**
DFT, RDFT
1D, 2D, 3D,…
batch

## No LAPACK equivalent for spectral methods

- **Medium size 1D FFT (1k—10k data points) is most common library call**
  applications break down 3D problems themselves and then call the 1D FFT library
- **Higher level FFT calls rarely used**
  FFTW *guru* interface is powerful but hard to used, leading to performance loss
- **Low arithmetic intensity and variation of FFT use make library approach hard**
  Algorithm specific decompositions and FFT calls intertwined with non-FFT code

# FFTX and SpectralPACK: Long Term Vision

**Numerical Linear Algebra**

**LAPACK**
LU factorization
Eigensolves
SVD
…

**BLAS**
BLAS-1
BLAS-2
BLAS-3

**Spectral Algorithms**

**SpectralPACK**
Convolution
Correlation
Upsampling
Poisson solver
…

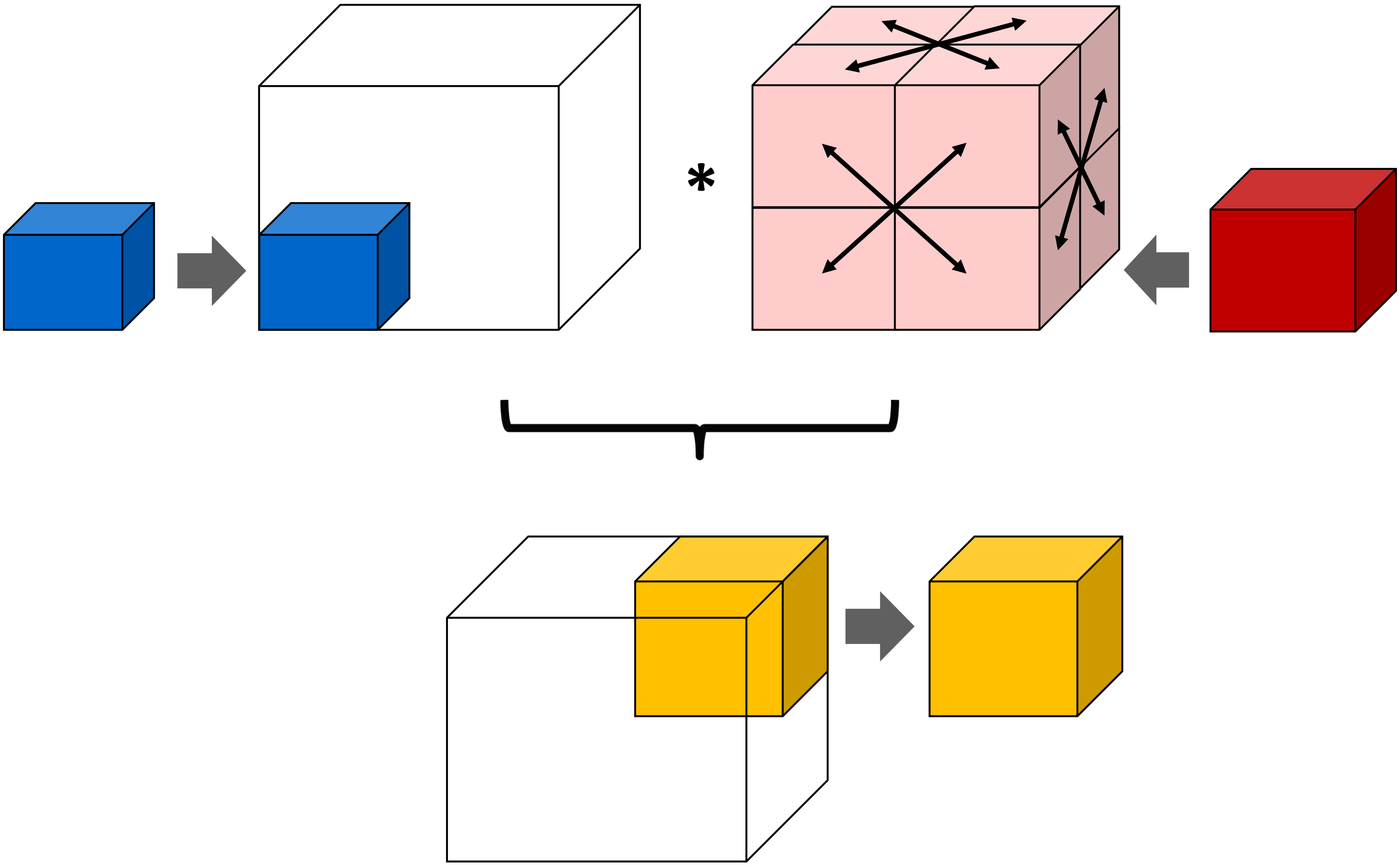
**FFTX**
DFT, RDFT
1D, 2D, 3D,…
batch

**Define the LAPACK equivalent for spectral algorithms**

- **Define FFTX as the BLAS equivalent**
  provide user FFT functionality as well as algorithm building blocks
- **Define class of numerical algorithms to be supported by SpectralPACK**
  PDE solver classes (Green's function, sparse in normal/k space,…), signal processing,…
- **Define SpectralPACK functions**
  circular convolutions, NUFFT, Poisson solvers, free space convolution,…

***FFTX and SpectralPACK solve the "spectral dwarf" long term***

# Example: Hockney Free Space Convolution

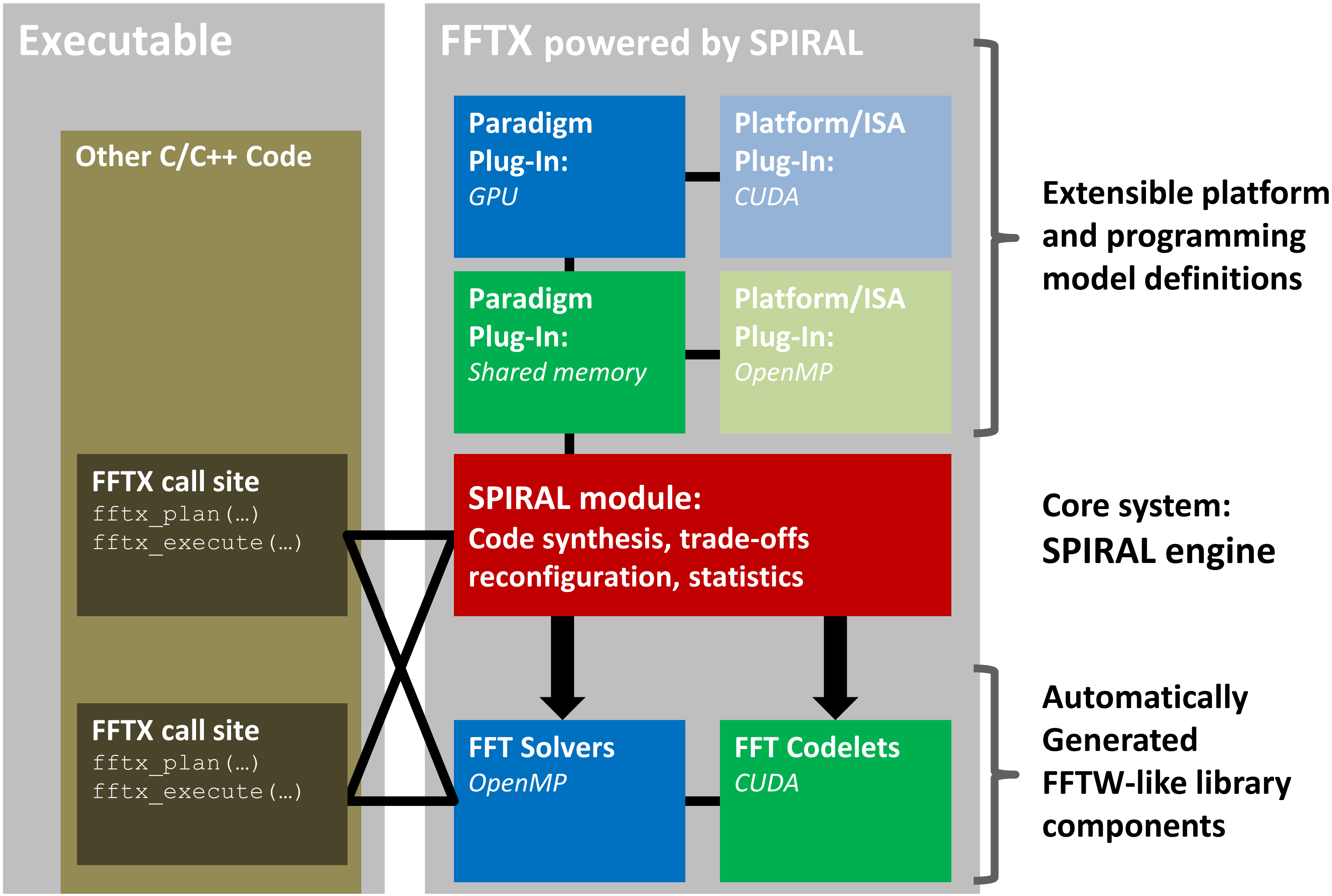


# Example: Hockney Free Space Convolution

```
fftx_plan pruned_real_convolution_plan(fftx_real *in, fftx_real *out, fftx_complex *symbol,
        int n, int n_in, int n_out, int n_freq) {
    int rank = 1,
    batch_rank = 0,
    ...
    fftx_plan plans[5];
    fftx_plan p;

    tmp1 = fftx_create_zero_temp_real(rank, &padded_dims);

    plans[0] = fftx_plan_guru_copy_real(rank, &in_dimx, in, tmp1, MY_FFTX_MODE_SUB);

    tmp2 = fftx_create_temp_complex(rank, &freq_dims);
    plans[1] = fftx_plan_guru_dft_r2c(rank, &padded_dims, batch_rank,
        &batch_dims, tmp1, tmp2, MY_FFTX_MODE_SUB);

    tmp3 = fftx_create_temp_complex(rank, &freq_dims);
    plans[2] = fftx_plan_guru_pointwise_c2c(rank, &freq_dimx, batch_rank, &batch_dimx,
        tmp2, tmp3, symbol, (fftx_callback)complex_scaling,
        MY_FFTX_MODE_SUB | FFTX_PW_POINTWISE);

    tmp4 = fftx_create_temp_real(rank, &padded_dims);
    plans[3] = fftx_plan_guru_dft_c2r(rank, &padded_dims, batch_rank,
        &batch_dims, tmp3, tmp4, MY_FFTX_MODE_SUB);

    plans[4] = fftx_plan_guru_copy_real(rank, &out_dimx, tmp4, out, MY_FFTX_MODE_SUB);

    p = fftx_plan_compose(numsubplans, plans, MY_FFTX_MODE_TOP);

    return p;
}
```

***Looks like FFTW calls, but is a specification for SPIRAL***

# FFTX Backend: SPIRAL



# Generated Code For Hockney Convolution

```
void ioprunedconv_130_0_62_72_130(double  *Y, double  *X, double * S) {
    static double D84[260] = {65.5, 0.0, (-0.5000000000001132), (-20.686114762237267),
      (-0.500000000000081), (-10.337014680426078), (-0.5000000000000455),
    ...
for(int i18899 = 0; i18899 <= 1; i18899++) {
    for(int i18912 = 0; i18912 <= 4; i18912++) {
        a9807 = ((2*i18899) + (4*i18912));
        a9808 = (a9807 + 1);
        a9809 = (a9807 + 52);
        a9810 = (a9807 + 53);
        a9811 = (a9807 + 104);
        a9812 = (a9807 + 105);
        s3295 = (*((X + a9807)) + *((X + a9809)) + *((X + a9811)));
        s3296 = (*((X + a9808)) + *((X + a9810)) + *((X + a9812)));
        s3297 = (((0.3090169943749474**((X + a9809))
            - (0.8090169943749474 5**((X + a9811)))) + *((X + a9807)));
        s3298 = (((0.3090169943749474**((X + a9810))
            - (0.8090169943749474 5**((X + a9812)))) + *((X + a9808)));
        s3299 = (((0.3090169943749474**((X + a9811))
            - (0.8090169943749474 5**((X + a9809)))) + *((X + a9807)));
        ...
        *((104 + Y + a12569)) = ((s3983 - s3987) + (0.8090169943749474 5*t6537)
            + (0.5877852522924731 4*t6538));
        *((105 + Y + a12569)) = (((s3984 - s3988) + (0.8090169943749474 5*t6538))
            - (0.5877852522924731 4*t6537));
    }
}
```

**FFTX/SPIRAL with OpenACC backend:**
**15 % faster than cuFFT expert interface**


***1,000s of lines of code, cross call optimization, etc., transparently used***

F. Franchetti, D. G. Spampinato, A. Kulkarni, D. T. Popovici, T. M. Low, M. Franusich, A. Canning, P. McCorquodale, B. Van Straalen, P. Colella: **FFTX and SpectralPack: A First Look**, IEEE International Conference on High Performance Computing, Data, and Analytics (HiPC), 2018

# Organization

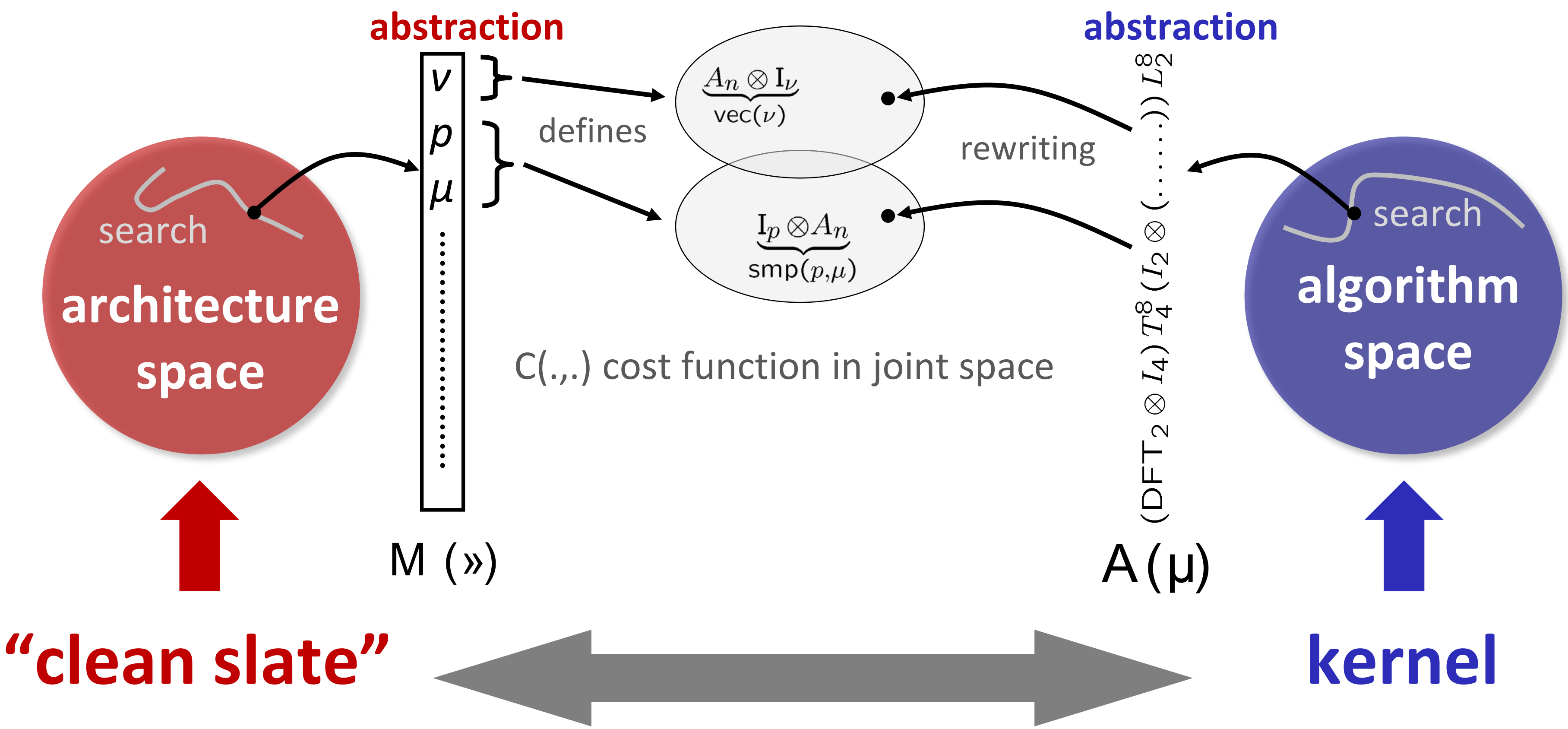
- SPL: Problem and algorithm specification
- Σ-SPL: Automating high level optimization
- Rewriting: Formal parallelization
- Rewriting: Vectorization
- Verification
- Spiral as FFTX backend
- **Summary**

# Co-Optimizing Architecture and Kernel

Architectural parameter:
Vector length,
#processors, …

**Model: common abstraction**
**= spaces of matching formulas**

Kernel:
problem size,
algorithm choice

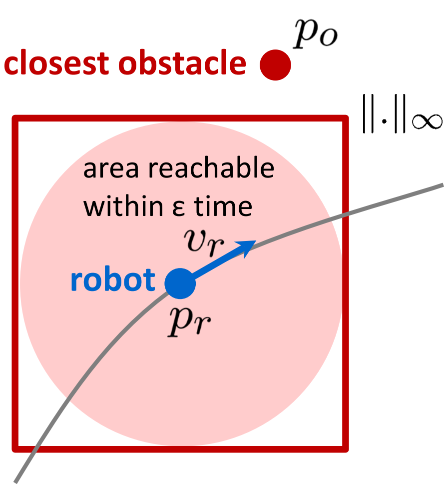


# High Assurance Spiral

## Equations of Motion

$$v_r = \dot{x}, \quad 0 \le v_r \le V$$
$$a = \dot{v}_r, \quad -b \le a \le A$$

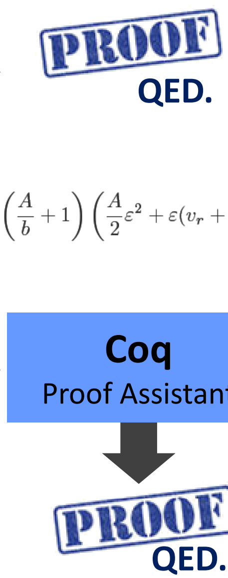
## Safety condition




$$\|p_r - p_o\|_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r+V)\right)$$

$$v_r = \dot{x}, \quad 0 \le v_r \le V$$
$$a = \dot{v}_r, \quad -b \le a \le A$$

**KeYmaera**
Hybrid Theorem Prover




$$\|p_r - p_o\|_\infty > \frac{v_r^2}{2b} + V\frac{v_r}{b} + \left(\frac{A}{b}+1\right)\left(\frac{A}{2}\varepsilon^2 + \varepsilon(v_r+V)\right)$$


**HA Spiral**
Code Synthesis

**Coq**
Proof Assistant

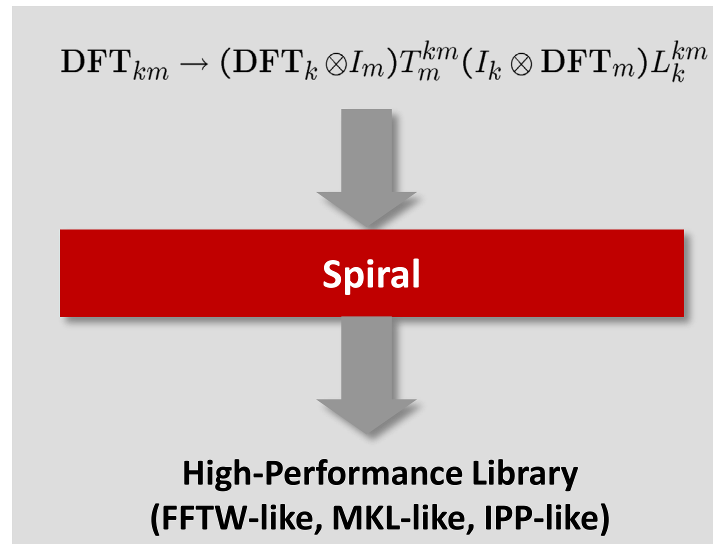
```
int dwmonitor(float  *X, double  *D) {
    __m128d u1, u2, u3, u4, u5, u6, u7, u8,...
    unsigned _xm = _mm_getcsr();
    _mm_setcsr(_xm & 0xffff0000 | 0x0000dfc0);
    u5 = _mm_set1_pd(0.0);
    u2 = _mm_cvtps_pd(_mm_addsub_ps(
    _mm_set1_ps(FLT_MIN), _mm_set1_ps(X[0])));
    u1 = _mm_set_pd(1.0, (-1.0));
    for(int i5 = 0; i5 <= 2; i5++) {
        x6 = _mm_addsub_pd(_mm_set1_pd((DBL_MIN
            +DBL_MIN)), _mm_loaddup_pd(&(D[i5])));
        x1 = _mm_addsub_pd(_mm_set1_pd(0.0), u1);
        x2 = _mm_mul_pd(x1, x6);
    ...
```


F. Franchetti, T. M. Low, S. Mitsch, J. P. Mendoza, L. Gui, A. Phaosawasdi, D. Padua, S. Kar, J. M. F. Moura, M. Franusich, J. Johnson, A. Platzer, and M. Veloso
**High-Assurance SPIRAL: End-to-End Guarantees for Robot and Car Control**
IEEE Control Systems Magazine, 2017, pages 82-103.

# General Size Library Generation

## Input:

- **Transform**: $\mathrm{DFT}_n$
- **Algorithms**: $\mathrm{DFT}_{km} \rightarrow (\mathrm{DFT}_k \otimes I_m) T_m^{km} (I_k \otimes \mathrm{DFT}_m) L_k^{km}$
  $\mathrm{DFT}_2 \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- **Vectorization**: 2-way SSE
- **Threading**: Yes



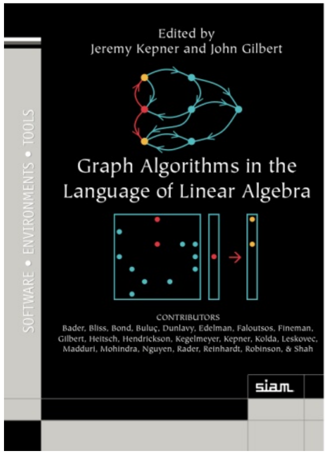

## Output:

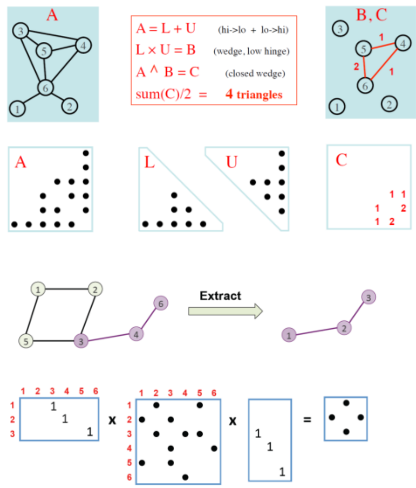
- Optimized library (10,000 lines of C++)
- For general input size
  (**not** collection of fixed sizes)
- Vectorized
- Multithreaded
- With runtime adaptation mechanism
- Performance competitive with hand-written code

Yevgen Voronenko, Frédéric de Mesmay and Markus Püschel
**Computer Generation of General Size Linear Transform Libraries**
Proc. International Symposium on Code Generation and Optimization (CGO), pp. 102-113, 2009

# Graph Algorithms in SPIRAL

## Foundation




## Context

- Kepner & Gilbert et al recast graph algorithms as linear algebra operations
- GraphBLAS Forum and reference implementations
- IBM, Intel, national labs etc. on board

## Formalization

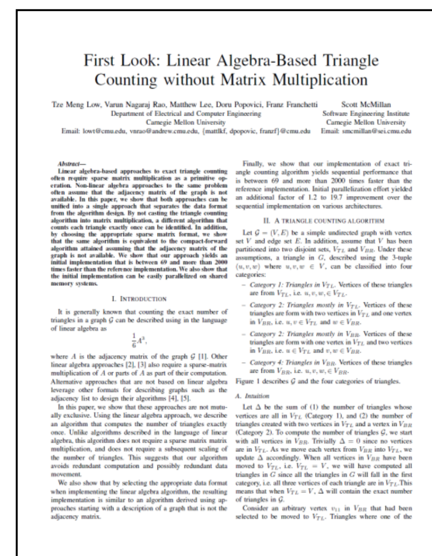
| Operation | Mathematical Description | Output | Inputs |
|---|---|---|---|
| mxm | $C\langle\neg M, z\rangle = C \odot (A^T \oplus.\otimes B^T)$ | C | ¬, M, z, ⊙, A, T, ⊕.⊗, B, T |
| mxv, (vxm) | $c\langle\neg m, z\rangle = c \odot (A^T \oplus.\otimes b)$ | c | ¬, m, z, ⊙, A, T, ⊕.⊗, b |
| eWiseMult | $C\langle\neg M, z\rangle = C \odot (A^T \otimes B^T)$ | C | ¬, M, z, ⊙, A, T, ⊗, B, T |
| eWiseAdd | $C\langle\neg M, z\rangle = C \odot (A^T \oplus B^T)$ | C | ¬, M, z, ⊙, A, T, ⊕, B, T |
| reduce (row) | $c\langle\neg m, z\rangle = c \odot [\oplus_j A^T(:,j)]$ | c | ¬, m, z, ⊙, A, T, ⊕ |
| apply | $C\langle\neg M, z\rangle = C \odot f(A^T)$ | C | ¬, M, z, ⊙, A, T, f |
| transpose | $C\langle\neg M, z\rangle = C \odot A^T$ | C | ¬, M, z, ⊙, A (T) |
| extract | $C\langle\neg M, z\rangle = C \odot A^T(i,j)$ | C | ¬, M, z, ⊙, A, T, i, j |
| assign | $C\langle\neg M, z\rangle (i,j) = C(i,j) \odot A^T$ | C | ¬, M, z, ⊙, A, T, i, j |
| build (meth.) | $C = \mathbb{S}^{m\times n}(i,j,v,\odot)$ | C | ⊙, m, n, i, j, v |
| extractTuples (meth.) | $(i,j,v) = A$ | i,j,v | A |

Notation: **i**,**j** – index arrays, **v** – scalar array, **m** – 1D mask, other **bold-lower** – vector (column), **M** – 2D mask, other **bold-caps** – matrix, **T** – transpose, ¬ - structural complement, z – clear output, ⊕ monoid/binary function, ⊕.⊗ semiring, blue – optional parameters, red – optional modifiers

## HIVE Graph Challenge


GraphChallenge DARPA HIVE amazon webservices

Graph Challenge Champions

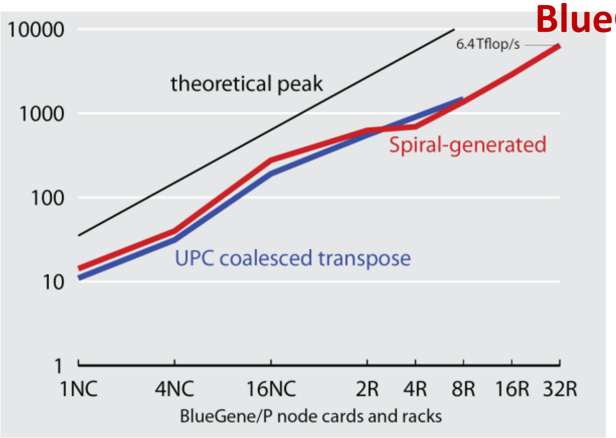

First Look: Linear Algebra-Based Triangle Counting without Matrix Multiplication

*In collaboration with CMU-SEI*

# SPIRAL FFTs in HPC/Supercomputing

- **NCSA Blue Waters**
  PAID Program, FFTs for Blue Waters
- **RIKEN K computer**
  FFTs for the HPC-ACE ISA
- **LANL RoadRunner**
  FFTs for the Cell processor
- **PSC/XSEDE Bridges**
  Large size FFTs
- **LLNL BlueGene/L and P**
  FFTW for BlueGene/L's Double FPU
- **ANL BlueGene/Q Mira**
  Early Science Program, FFTW for BGQ QPX

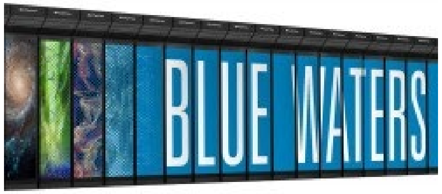
**Global FFT (1D FFT, HPC Challenge)**
**performance [Gflop/s]**




**6.4 Tflop/s on BlueGene/P**

**BlueGene/P at Argonne National Laboratory**
128k cores (quad-core CPUs) at 850 MHz



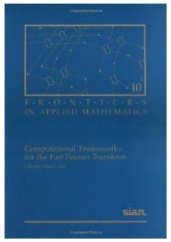


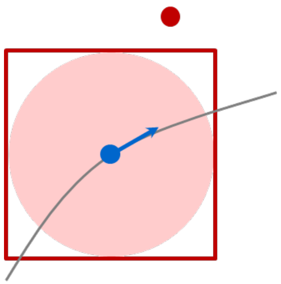
***2006 Gordon Bell Prize (Peak Performance Award) with LLNL and IBM***

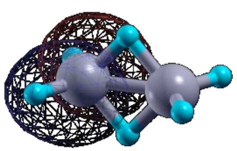
***2010 HPC Challenge Class II Award (Most Productive System) with ANL and IBM***

# Summary: Formal Code Synthesis

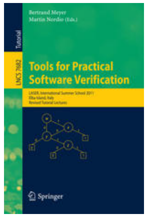
## Algorithms

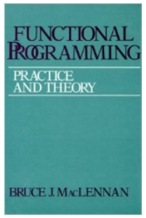


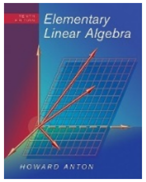





## Correctness




```
int dwmonitor(float  *X, double  *D) {
    __m128d u1, u2, u3, u4, u5, u6, u7, u8,...
    unsigned _xm = _mm_getcsr();
    _mm_setcsr(_xm & 0xffff0000 | 0x0000dfc0);
    u5 = _mm_set1_pd(0.0);
    u2 = _mm_cvtps_pd(_mm_addsub_ps(
    _mm_set1_ps(FLT_MIN), _mm_set1_ps(X[0])));
    u1 = _mm_set_pd(1.0, (-1.0));
    for(int i5 = 0; i5 <= 2; i5++) {
        x6 = _mm_addsub_pd(_mm_set1_pd((DBL_MIN
            +DBL_MIN)), _mm_loaddup_pd(&(D[i5])));
        x1 = _mm_addsub_pd(_mm_set1_pd(0.0), u1);
        x2 = _mm_mul_pd(x1, x6);
    ...
```


## Hardware



# More Information:

www.spiral.net

www.spiralgen.com

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## Overview Papers

F. Franchetti, T. M. Low, D. T. Popovici, R. M. Veras, D. G. Spampinato, J. R. Johnson, M. Püschel, J. C. Hoe, J. M. F. Moura: **SPIRAL: Extreme Performance Portability,** Proceedings of the IEEE, Vol. 106, No. 11, 2018.
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