Chapter 4

Integrated Micromechanical Testbed

4.1 Introduction

Digital closed-loop control of MEMS involves the design and simulation of mixed analog-digital, electro-mechanical systems. Prior work has focussed exclusively on the sigma-delta ($\Sigma-\Delta$) stabilization of single-mode microaccelerometers [88, 64, 89]. MEMS applications such as vibratory rate gyroscopes, multi-axis accelerometers and actively positioned mirrors require digital control of multiple degrees of freedom. In this chapter, we present an integrated testbed that enables the experimental verification of models and simulations of general control strategies for a suspended plate with three degrees of freedom.

The testbed project closely followed work at U. C. Berkeley on a force-balanced $\Sigma-\Delta$ accelerometer by Yun [64]. In particular, the fabrication process, capacitive position sensor, and two-bit$^1$ feedback concepts are borrowed from his accelerometer design. Based on this previous work, we have made improvements in all three of these areas, with the details described in sections 4.4, 3.6.1, and 5.5.

In this chapter, we will first discuss the $\Sigma-\Delta$ architecture. Then, the testbed design will be presented, followed by a description of the fabrication process. Modeling, simulation, and experimental results are presented in the following chapters.

$^1$That is, in the digital sense.
Figure 4.1: Block diagram of a conventional \( \Sigma-\Delta \) modulator.

## 4.2 Sigma-Delta Architecture

An understanding of micromechanical \( \Sigma-\Delta \) control requires an introduction to the conventional second-order \( \Sigma-\Delta \) modulator [90], shown in Figure 4.1. The input data, \( x(nT) \), is sampled with period, \( T \), and passed through a closed-loop system that has two discrete integrators and a one-bit analog-to-digital (A/D) converter in the forward path. The A/D output, \( y(nT) \), is converted to an analog signal (D/A conversion) and fed back to the summing nodes in front of the integrators. If the sampling rate is much higher than the input signal bandwidth, the digital output will be a pulse-density-modulated version of the input signal. Subsequent digital low-pass filtering of the output bitstream provides a precision A/D conversion of the input signal.

Advantages of the \( \Sigma-\Delta \) architecture over other A/D-converter designs include superior linearity, large dynamic range, and simple electronics. The input-to-output relationship is inherently linear, because the D/A feedback signal has only two states. The integrators attenuate the high-frequency switching component of the feedback, reducing the switching effects on the low-frequency output signal. Because the feedback is quantized into two states, unwanted noise from these high-frequency switching components is called quantization noise. Output precision is dependent on timing precision, not on the analog precision of the components in the loop. Essentially, crude two-state precision in the feedback is compensated by having excellent temporal resolution. Thus, the \( \Sigma-\Delta \) loop can be implemented with fast, accurate digital timing circuitry that is easy to construct.

Sigma-delta feedback was first applied to micromechanical elements in a bulk-micromachined accelerometer by Henrion [88]. In that paper, the basic concepts of the micromechanical \( \Sigma-\Delta \) loop were introduced. The testbed \( \Sigma-\Delta \) loop, shown in Figure 4.2, is
Figure 4.2: Block diagram of a micromechanical $\Sigma$-$\Delta$ loop.

Based on a high-frequency one-bit comparison between the output of a position sensor and an external reference, which generates a serial bitstream. An electrostatic actuator converts the bitstream into two force-feedback levels that act on the structure. The benefits of $\Sigma$-$\Delta$ control extend to the micromechanical system, providing inherently linear, precision sensing of the external force input, without the need for precision position sensing. The micromechanical mass-spring-damper replaces the two discrete-time integrators in the conventional $\Sigma$-$\Delta$ modulator. Frequency components of the feedback above mechanical resonance are attenuated by the mechanical double integration. With proper attention to stability of the digital control loop, the mass is forced to the average position corresponding to the reference input. The critical circuit design issues are the position-sensor sensitivity, (not precision) and speed to reduce the quantization position noise. An analysis of quantization noise will be presented later, in section 5.6.2.

Unlike the conventional $\Sigma$-$\Delta$ modulator, the micromechanical version has no inner-feedback loop. In mechanically underdamped operation, direct feedback from the $\Sigma$-$\Delta$ comparator provides a force which is $180^\circ$ out of phase with the displacement, resulting in uncontrolled oscillation at the resonant frequency. Thus, some other form of compensation is necessary to implement velocity-proportional feedback that would stabilize an underdamped micromechanical mass-spring-damper. With compensation, the displacement stabilizes to a bounded limit-cycle oscillation. Limit cycles in conventional $\Sigma$-$\Delta$ modulators have been studied previously to determine the structure of quantization noise [91].

Details of the digital compensation and an analysis of limit cycles are presented in
Figure 4.3: Block diagram of a micromechanical $\Sigma$-$\Delta$ loop with digital compensation in the feedback path.

Section 5.5; we will only provide a brief description of the compensation here. Previously, analog lead-lag compensation was used in the Henrion accelerometer, to permit operation in a vacuum. Two-bit digital compensation of a high-$Q$ micromechanical $\Sigma$-$\Delta$ loop was first proposed by Yun [64]. A similar digital feedback scheme is implemented in the testbed, but, instead of a fixed two-bit gain, an adjustable gain can be set to minimize the limit-cycle amplitude. A digital compensation circuit in the feedback path drives a differential electrostatic actuator, as shown in Figure 4.3. The extra high-force levels are introduced to implement digital velocity-proportional feedback. Sensor linearity is not affected, because the high-force pulses average to zero. Previous compensation schemes were presented without experimental verification. We have successfully operated the testbed at low pressures, with the results reported in section 6.9.

4.3 Testbed Description

A rendering of one quarter of the testbed is shown in Figure 4.4. A polysilicon plate, which measures 360 $\mu$m $\times$ 380 $\mu$m $\times$ 1.6 $\mu$m in thickness, is suspended 2.2 $\mu$m above the substrate by four serpentine springs. The springs passively constrain lateral motion of the plate. Spring compliances can be configured in seven different ways\(^2\) by

\[^2\text{The serpentine spring layout is shown in Figure A.3 followed by a tabulation of spring constants and resonant frequencies in Table A.2.}\]
selectively cutting fuses\textsuperscript{3} that anchor the springs in several locations. Physical dimensions and other parameters of the testbed are listed in appendix A.1. The plate is overdamped at room-ambient pressure; damping can be reduced to nearly zero by reducing the pressure. Vertical position, $z$, and angular rotation about the lateral axes, $\theta$ and $\phi$, are controlled by four digital feedback loops. A parallel-plate vertical-position sense capacitor is located under each quadrant of the plate, forming a bridge with a fixed reference capacitor. Use of a driven shield electrode together with an integrated CMOS buffer amplifier, are required to detect the voltage change on the high-impedance node of the capacitive bridge. Integration of the buffer amplifiers eliminates most of the interconnect parasitic capacitance that would cause signal degradation. Off-chip $\Sigma$-$\Delta$ electronics supply digital feedback to electrostatic actuators at each corner of the plate.

The testbed is part of the multi-project chip shown in Figure 4.5. Microsystem designs from several other micromechanical research projects at U. C. Berkeley share area on the 1 cm $\times$ 1 cm die, including an accelerometer (designed by Crist Lu and Weijie Yun), gyroscopes (Per Ijung), microresonator oscillators (Clark Nguyen), microposition

\textsuperscript{3} A close-up of one of the fuses is shown in Figure 2.45.
Figure 4.5: Optical photograph of the multi-project MICS chip.
controllers (Patrick Cheung), and electrostatic voltmeters (Kai Chen and David Loconto). The remaining real-estate is used for process monitors and test structures.

Figure 4.6 is an optical micrograph of the testbed, showing the micromechanical elements integrated with CMOS circuitry. Reference capacitors are made from rigidly supported micromechanical structures which match the sense capacitor layout. Variable gain differential amplifiers, included in the layout, suffer from a design error and are not used in the system. The eight reference capacitors located at the upper and lower corners of the layout are associated with the differential amplifiers, and are also not used. Bias circuitry is shared by the four buffer amplifiers. An expanded view of the polysilicon plate is shown in Figure 4.7. Two comb drives, located on two opposing sides of the plate, provide both a vertical levitation force and a lateral force for displacements up to 10 μm. Each comb has 39, 2 μm-wide, 10 μm-long fingers. The four capacitive position sensors underneath the plate are identified, along with the upper parallel-plate feedback actuators.

A block diagram of one of the four independent feedback channels is shown in Figure 4.8. Each movable sense capacitor, $C_s$, forms a voltage divider with a corresponding fixed reference capacitor, $C_r$. Balanced square-wave voltage signals, $V_{m+}$ and $V_{m-}$, provide up to 100 kHz, ±300 mV modulation across the divider. A minimum-area diode,
Figure 4.7: Expanded view of the polysilicon plate and serpentine springs. The white dashed rectangle delimits the area shown in Figure 4.10.

Figure 4.8: Block diagram of one corner of the integrated testbed.
connected to the buffer input, supplies a dc bias to ground. The buffer output, $V_{\text{sense}}$, is routed off-chip, demodulated, and amplified to generate a signal proportional to the peak-to-peak buffer output, $V_{pp}$. This signal is compared with an external position reference voltage, $V_{\text{ref}}$, quantizing the plate position into two states. The external reference enables multi-mode position control, self-test, and offset trim. The comparator output drives a level-shifting network, which generates feedback voltages containing a difference term to stabilize underdamped mechanical systems. Feedback voltage levels are adjustable, allowing experimentation with values of full-scale force and compensation. The feedback voltages are sent on-chip to the parallel-plate actuators, $C_{\text{up}}$ and $C_{\text{down}}$. Digital bitstreams from the comparators are filtered and combined to form multi-mode acceleration sense signals. More detailed schematics and photographs of the external electronics are in appendix A.2.1.

### 4.4 Fabrication

Fabrication of the integrated testbed utilizes the Berkeley MICS process: p-well 3-μm CMOS integrated with polysilicon microstructures [64,92]. A MICS process cross-section is shown in Figure 4.9. We will highlight some of the processing features and issues in this section. A complete MICS process flow can be referenced in appendix C.

#### 4.4.1 Second Structural Polysilicon Layer

We have extended the original MICS process to include a second structural polysilicon layer (sensor poly-3 in Figure 4.9). The upper feedback actuator is made from the
second structural layer, and shown in Figure 4.10. Upper limit stops, which prevent shorting of the actuators, are also made from the second structural layer. The spacer PSG acts as passivation for the tungsten metallization, and is only removed around the microstructure areas. Microstructure release in 5:1 BHF is done in darkness to avoid electrochemical etching of the polysilicon bonding pads.

Our original attempts at forming the second structural layer resulted in unwanted polysilicon stringers being deposited in 2 µm-wide gaps between structures in the first structural layer (poly-2). The SEM of a comb-finger cross-section in Figure 4.11 shows the poly-3 stringers lodged between the poly-2 comb fingers. The phosphosilicate glass (PSG) film that acts as a vertical spacer between poly-2 and poly-3 does not conformally coat the region in the gap; instead, a keyhole-shaped void is formed. Subsequent deposition of the poly-3 layer fills this void from the open ends, forming a stringer. Isotropic etching will not remove the stringers, since they are covered with the PSG spacer film. In order to avoid the stringer formation, the sacrificial layer underlying the poly-3 layer is planarized by using a spin-on glass (SOG) film to fill the gaps. Processing the 2.7 µm-thick sacrificial layer involves several steps: a 7000 Å-thick PSG deposition, two coats of spin-on glass (SOG) film and curing up to 425°C, etch-back of the SOG, a 2 µm-thick PSG deposition, and rapid-thermal annealing (RTA) at 900°C for 30 s to densify the PSG. We observe some
cracking and browning of the SOG film after the RTA. The cracks are of sub-$\mu$m width and encased in PSG, therefore, they do not cause problems in subsequent processing steps. After sacrificial release, however, very small (< 2000Å-wide) polymer stringers, shown in Figure 4.12, appear at the former SOG-crack locations, which are possibly formed from residual solvent left in the SOG and baked at 900°C. We are able to remove the stringers with a 1 hour, 5:1 $\text{H}_2\text{SO}_4:\text{H}_2\text{O}_2$ (pirahna) soak immediately after the BHF release.

4.4.2 Rapid-Thermal Annealing of Residual Stress

As-deposited highly-doped polysilicon films have a large compressive residual stress and a large stress gradient through the film thickness, causing microstructures to buckle and deflect out of plane. Residual stress and stress gradients are greatly reduced by annealing the polysilicon film at temperatures above 1000°C for 1 hour [20]. The CMOS circuits and contacts under our microstructures cannot survive this long of an annealing step. Instead, we substitute an RTA step in the MICS process to anneal the structural polysilicon layers.

We performed a simple experiment to determine the optimum RTA time and tem-
temperature to achieve minimum film stress. We deposited 1.87 μm of in situ doped polysilicon (12 hr deposition at 610°C, 375 mT, 100 sccm SiH₄, 1 sccm PH₃) on top of 2 μm PSG, and then patterned microstructural strain gauges. The wafer was cleaved into 4-die samples, each sample was subjected to a different RTA cycle, and all samples were simultaneously released in 10:1 HF. Our strain gauge design, pictured in Figure 4.13, is similar to the design reported by Lin [93]. Upon microstructure release, the 1 mm-long, 16 μm-wide polysilicon beam in the strain gauge will release its residual stress by expanding, assuming compressive residual stress. The beam exerts a force on the crab-leg pivot, and the resulting angular deflection is magnified 18× by a cantilevered pointer. Deflection at the end of the pointer is measured optically using a displacement vernier (shown in Figure 4.13(b)) and the results are converted to strain. A summary of experimental results are presented in Table 4.1. Although the measurement uncertainty is large for the 1050°C anneals, we can draw some conclusions from this data. Values of residual stress are reduced more than 20× below the as-deposited value of 289 MPa. Microstructure stress gradients are not observable for RTA below 950°C, except on the sample annealed at 2 min, 950°C. We chose to anneal our MICS wafers twice at 900°C for 30 s, which is the same RTA cycle used to densify the PSG films. Therefore, the first structural polysilicon layer is annealed with three RTA cycles at 900°C. The resulting structural polysilicon is very flat, with no observable buckling, as illustrated
Figure 4.13: Strain-gauge design used to measure residual stress in microstructural polysilicon films. (a) Layout showing the 1 mm-long, 16 μm-wide beam that releases its built-in axial stress. Crab-leg tethers and dimples keep the beam from sticking to the substrate without significantly affecting the strain measurement. (b) Enlargement of the displacement vernier with 0.2 μm resolution.

<table>
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<tr>
<th>temperature [°C]</th>
<th>time [min]</th>
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<th>stress (E=165GPa) [MPa]</th>
<th>curl (see caption)</th>
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<tr>
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<td>0.5</td>
<td>66±13†</td>
<td>11±2†</td>
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</tr>
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<td>850</td>
<td>1</td>
<td>73±20†</td>
<td>12±3†</td>
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</tr>
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<td>850</td>
<td>2</td>
<td>47±16†</td>
<td>8±3†</td>
<td>0</td>
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<td>49±36</td>
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<td>63±16</td>
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<tr>
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<td>33±95</td>
<td>5±16</td>
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<tr>
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<td>1</td>
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<td></td>
<td>1750±16</td>
<td>289±3</td>
<td>++</td>
</tr>
</tbody>
</table>

Table 4.1: Experimental results of *in situ* phosphorous-doped polysilicon stress annealing using RTA. Stress and strain are given with ±3-sigma bounds. Out-of-plane curl of 940 μm-long, 2 μm-wide cantilever beams is qualitatively measured with an optical microscope, where “0” is no observable curl, “+” is up to 1 μm of vertical deflection at the cantilever tip, and “++” is more than 1 μm of deflection at the cantilever tip. † Some fixed-fixed beams are buckled on the samples annealed at 850°C, casting doubt on the accuracy of these strain values.
in Figure 4.10.

### 4.4.3 TiSi₂ Contacts

Tungsten interconnect with TiN/TiSi₂ contacts is necessary to withstand the temperature of subsequent microstructure processing steps. The original MICS process used a 600 Å-thick sputtered Ti layer followed by a 600°C, 30 s RTA in N₂ to form the TiSi₂ regions in the contacts [83]. After the excess Ti is stripped in 3:1 NH₄OH:H₂O₂, another RTA (1000 °C, 30 s, N₂) is performed which lowers the contact resistance by creating stoichiometric TiSi₂. Tungsten metallization is then sputtered and patterned. A thin TiN layer formed during the second RTA acts as a diffusion barrier to tungsten during the subsequent 835°C silicon-nitride deposition.

We found that the Ti reaction with Si consumed much of the boron in the p⁺ regions, producing high-resistance non-ohmic contacts. We modified the process by reducing the Ti thickness to 250 Å, lowering the second RTA temperature to 850°C, and including a 250 Å-thick reactively sputtered TiN layer on top of the TiSi₂. The thicker diffusion barrier allows deposition of a thick, low-stress silicon-nitride film at 835°C to effectively protect the circuits from HF attack during the microstructure release step. Measured contact resistance is 27 Ω for n⁺ contacts at currents under 100 μA, and 55 Ω for p⁺ contacts at currents under 2 mA. For both kinds of contacts, the sample standard deviation is about 5 Ω across the wafer. When larger currents are passed through the contacts, the resistance drops to about half the low-current value.

### 4.4.4 Testbed Fabrication Results

One wafer, the cmos30-2 wafer⁴, survived the 20-mask MICS process flow and provided working testbed devices. The end-product is not without some flaws, however. There are problems with HF attack of the silicon-nitride passivation over and near tungsten interconnect, resulting in open tungsten-polysilicon contacts in some regions. In particular, the sense and reference capacitors in the testbed position sensors are disconnected from the buffer on many samples. Unfortunately, the connection to the reference capacitors

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⁴Wafers originating from the Berkeley baseline CMOS process are numbered by run and wafer in the lot. The cmos30-2 wafer is the second wafer out of the 30th baseline run in the microlab. Specific process information on each run is accessible by computer from the Berkeley Microfabrication Laboratory process logs.
has a yield close to zero, so the position sensors resemble the single-capacitor sensor of section 3.6.1.1.

One device was packaged in a 24-pin dual-inline package (DIP) for testing. Pin assignments are supplied in Table A.4. This device has three working position sensors, located at the upper-right, upper-left, and lower-left quadrants. A resistive short between the input sense line and the shield has rendered the lower-right position sensor inoperable. Only the first fuse is cut from the serpentine springs, providing a vertical spring constant of 0.25 N/m and a resonant frequency of 3.7 kHz. We did not have enough samples to study the testbed with more compliant spring configurations.

\[5\text{However, the fundamental mode is not in the vertical direction. The }\theta\text{-rotational mode has a resonant frequency of }2.9\text{ kHz.}\]
Chapter 5

System Simulation and Analysis

5.1 Introduction

Our approach to system simulation of the testbed uses the lumped-parameter components introduced in chapter 3. Groundwork for the simulations requires identification and construction of the component macro-models. We have made macro-models compatible with two simulation tools: SPICE and MATLAB$^\text{TM}$ [12]. Low-level simulation of circuits with microstructures is best done using SPICE, but the simulations are slow, and time-step convergence can be problematic. Tools for numerical computation, such as MATLAB, can be used to provide a faster, high-level simulation, using behavioral macro-models.

In the last sections of this chapter, two topics specific to micromechanical $\Sigma$-$\Delta$ loops, limit cycles and equivalent noise in the acceleration signal, will be analyzed. High-frequency limit cycles arise from the digital feedback force imposed on the testbed, causing chatter in the plate’s position. In particular, loop compensation is needed to bound limit cycles under mechanically underdamped conditions. Dynamic range of the output acceleration signal, generated by digitally filtering the $\Sigma$-$\Delta$ bitstreams, is affected by several noise sources, including Brownian noise, quantization noise, electronic noise, and interconnect noise.

5.2 System Modeling

The micromechanical elements of the testbed, shown in simplified layout form in Figure 5.1, can be partitioned into several kinds of components: a rigid body (the suspended
plate), electrostatic actuators, and capacitive position sensors. Models of these components are then incorporated into the testbed simulations. We will simulate behavior of the testbed in the vertical direction, $z$, $\theta$, and $\phi$, but lateral motion will be ignored. Actuator modeling will be restricted to the parallel-plate feedback actuators; the comb drives are turned off in the simulations. In all of our experiments, the testbed suspension is configured with one fuse cut on each spring. Therefore, we will present modeling and simulation results for that configuration, unless otherwise specified. A summary of model parameter definitions and values for the testbed are given in appendix A.1 for reference.

5.2.1 Testbed Plate Model

The resonant modes of the testbed plate are determined from a linear finite-element analysis [53], using 3-node quadratic beam elements for the springs and 9-node quadrilateral shell elements for the plate. Figure 5.2 displays the lowest eight modes for the plate, where each spring is configured with one fuse cut. Resonant frequency values of the lowest four modes, $\theta$-rotation (rotation about the $x$ axis), $z$-translation, $\phi$-rotation (rotation about the $y$ axis), and $x$-translation are between 2.8 kHz and 3.9 kHz. The other two rigid-body modes of the plate, $y$-translation and $\psi$-rotation (rotation about the $z$ axis), have slightly larger values of resonant frequency. Plate bending modes and spring-vibration modes have resonant frequency values that are over 24 times higher than the fundamental mode.

There are three controllable mechanical modes of the plate: $z$-translation, and the two rotations, $\theta$ and $\phi$, about the lateral axes. These modes are modeled with the rigid-body, mass-spring-damper equations of motion introduced in section se:equations-of-motion. Transfer functions for the modes are obtained by taking the Laplace transform of Equations (3.18)–(3.20) yielding

$$H_z(s) = \frac{\Delta z(s)}{F_z(s)} = \frac{1}{m(s^2 + 2\zeta_z \omega_z s + \omega_z^2)} \quad (5.1)$$

$$H_\theta(s) = \frac{\theta(s)}{\tau_\theta(s)} = \frac{1}{I_\theta(s^2 + 2\zeta_\theta \omega_\theta s + \omega_\theta^2)} \quad (5.2)$$

$$H_\phi(s) = \frac{\phi(s)}{\tau_\phi(s)} = \frac{1}{I_\phi(s^2 + 2\zeta_\phi \omega_\phi s + \omega_\phi^2)} \quad (5.3)$$

where the vertical displacement, $\Delta z$, is defined relative to the fabricated spacer gap of 2.2 $\mu$m.

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1A list of spring constants and resonant modes for each spring configuration is given in Table A.2.
Figure 5.1: Simplified layout of the testbed, where the plate's suspension is configured with one fuse cut on each spring.
<table>
<thead>
<tr>
<th>Mode 1: $\theta$, 2860 Hz</th>
<th>Mode 2: $z$, 3680 Hz</th>
</tr>
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<table>
<thead>
<tr>
<th>Mode 3: $\phi$, 3700 Hz</th>
<th>Mode 4: $x$, 3900 Hz</th>
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<table>
<thead>
<tr>
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<th>Mode 6: $\psi$, 8110 Hz</th>
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<tbody>
<tr>
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<td><img src="image6" alt="Mode 6" /></td>
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<table>
<thead>
<tr>
<th>Mode 7: plate, 69400 Hz</th>
<th>Mode 8: springs, 75500 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image7" alt="Mode 7" /></td>
<td><img src="image8" alt="Mode 8" /></td>
</tr>
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</table>

Figure 5.2: Illustration of the eight lowest mechanical modes of the testbed, where each spring is configured with one fuse cut.
The testbed system is analyzed by transforming the four quad-symmetric feedback loops into the three modes. However, since there are four feedback loops and only three controllable modes, the system is over-constrained. Offsets must be precisely adjusted to avoid undesired oscillations or force limiting in one of the loops. If the feedback signal levels were continuous, any position offsets would have resulted in four stable states for the plate position. With the quantized-state feedback, however, the electrostatic actuators will not limit as long as the position offsets are less than the amplitude of the limit cycles inherent in the system.

### 5.2.2 Position-Sensor Model

The testbed can control the plate position with a large range of motion relative to the gap spacing. An accurate model of the position sensor is needed to relate each position reference input to the vertical displacement. A layout of the position sensor electrodes under the plate is shown in Figure 5.3. We will use the quadrant numbering convention in Figure 5.3 throughout our discussion of the testbed. Numbering starts with “1” at the upper-right quadrant and progresses counter-clockwise to “4” at the lower-right quadrant. We will use the symbol $i$ as the quadrant index in equations. The square plate electrodes have dimensions $L_{sx,i} = L_{sy,i} = 100 \, \mu m$ and are located a distance $|x_{so,i}| = |y_{so,i}| = 60 \, \mu m$ from the centerlines of the plate.

The position sensor model has inputs of plate displacement and rotation, from which the sensor output voltage, force, and torque are calculated. In our first-order model, we assume a constant air gap across the sensor electrode area, and use a small-angle approximation. Because of their modular integration, models that incorporate the second-order effects of plate tilt can be easily substituted into the simulation. The air gap for each sensor is calculated at the center of the electrodes.

$$z_{s,i} = z_0 + \Delta z + y_{so,i} \theta - x_{so,i} \phi$$

(5.4)

In SPICE, the capacitive divider and unity-gain buffer amplifier, discussed in section 3.6.1.2, can be simulated directly to obtain the sensor output voltage; in MATLAB simulations, the sensor output voltage is calculated from Equation (3.178). A force and torque from the parallel-plate sensor electrodes act on the plate, and are given by

$$F_{sz,i} = - \frac{\alpha_i e A_{s,i}}{z_{s,i}^2} (V_{m,i} - V_{s,i})^2$$

(5.5)
Figure 5.3: Simplified testbed layout, identifying the position-sensor electrodes and lower-actuator electrodes.
\[
\begin{align*}
\tau_{s0,i} &= y_{so,i} F_{sz,i} \\
\tau_{s1,i} &= -x_{so,i} F_{sz,i}
\end{align*}
\]

where \( A_s \equiv L_{sx,i} L_{sy,i} \), and \( \alpha \) is a form factor that compensates for fringing around etch holes.

### 5.2.3 Electrostatic Actuator Model

The parallel-plate actuators, located at the plate’s corners, have electrodes above and below the plate to provide a differential force. Figure 5.4 shows the size and location of the upper actuators. The square plate electrodes have dimensions \( L_{ux,i} = L_{uy,i} = 54 \, \mu\text{m} \) and are located a distance \( |x_{uo,i}| = |y_{uo,i}| = 150 \, \mu\text{m} \) from the centerlines of the plate. The lower-actuator electrodes, identified in Figure 5.3, are located directly below the upper actuators, and are 2 \( \mu\text{m} \) smaller on each side. The displacement, force, and torque equations are similar to those for the sense electrodes. For the upper-actuator,

\[
\begin{align*}
z_{u,i} &= z_{uo} + \Delta z + y_{uo,i} \theta - x_{uo,i} \phi \\
F_{uz,i} &= \frac{\alpha_u \epsilon A_{u,i}}{z_{uo,i}^2} (V_{m,-} - V_{uo,i})^2 \\
\tau_{u0,i} &= y_{uo,i} F_{uz,i} \\
\tau_{u1,i} &= -x_{uo,i} F_{uz,i}
\end{align*}
\]

and for the lower-actuator,

\[
\begin{align*}
z_{l,i} &= z_{lo} + \Delta z + y_{lo,i} \theta - x_{lo,i} \phi \\
F_{lz,i} &= -\frac{\alpha_l \epsilon A_{l,i}}{z_{lo,i}^2} (V_{m,-} - V_{down,i})^2 \\
\tau_{l0,i} &= y_{lo,i} F_{lz,i} \\
\tau_{l1,i} &= -x_{lo,i} F_{lz,i}
\end{align*}
\]

where \( z_{uo} \) is the fabricated air-gap between the plate and the upper actuator. We have again introduced form-factors, \( \alpha_u \) and \( \alpha_l \), to compensate for the fringe fields from etch holes in the plates.

### 5.3 SPICE Simulation

We perform low-level simulation of microstructures with circuits using a commercial version of SPICE, HSPICE [3]. Each of the models described above is implemented as a
Figure 5.4: Simplified testbed layout, identifying the upper-actuator electrodes.
subcircuit. Mechanical interactions are communicated through node voltages in SPICE. By maintaining modular subcircuit models, we can improve the models without affecting the overall simulation. HSPICE input files of the testbed simulation and the subcircuit models are listed in appendix B.

5.3.1 Equivalent Circuit Model of a Mass-Spring-Damper

A mechanical mass-spring-damper can be modeled in SPICE by making an electrical analogy to an inductor-capacitor-resistor (LCR) circuit. Either the series-connected or parallel-connected LCR circuit can model the second-order mechanical system; we have chosen to use the series-connected LCR circuit shown in Figure 5.5. Each mechanical parameter is represented by an equivalent electrical parameter.

\[
\begin{align*}
L[m] &= A m \\
R[B] &= A B \\
C[k] &= \frac{1}{A k} \\
V[F] &= A F \\
V_c &= A k z \\
V[z] &= A z
\end{align*}
\]

The scaling factor, \( A \), is needed to keep the electrical impedance values of the LCR circuit from being very small, thereby avoiding convergence problems in SPICE. A useful value
for $A$ is $10^6$ for micromechanical components, producing the following parameter values, typical in the testbed operation:

\[
\begin{align*}
\text{Mechanical parameters:} & & \text{Electrical parameters:} \\
m = 4.7 \times 10^{-10} \text{ kg} & & L_z [m] = 0.47 \text{ mH} \\
k_z = 0.25 \text{ N/m} & & C_z [k_z] = 4.0 \mu \text{F} \\
B_z = 2.6 \times 10^{-4} \text{ N-s/m} & & R_z [B_z] = 260 \Omega \\
F_z = 100 \text{ nN} & & V [F_z] = 0.1 \text{ V} \\
z = 1.0 \mu \text{m} & & V [z] = 1 \text{ V}
\end{align*}
\]

### 5.3.2 Time-Varying Capacitor Model

A time-varying capacitor (TVC) subcircuit model for SPICE is shown in Figure 5.6. The TVC I-V characteristic is given by

\[
I = \frac{d}{dt}(C_s V_c V) \tag{5.22}
\]
where $C_s$ is a capacitance scaling factor, and $V_c$ is a voltage that specifies the instantaneous scaled capacitance value\(^2\). Since micromechanical capacitor values are usually between 1 fF and 1 pF, a good value for the capacitance scaling factor is 1 pF. The TVC subcircuit can be implemented in any version of SPICE that supports nonlinear voltage-controlled voltage sources.

5.3.3 Micromechanical Capacitor Model

We model both the actuators and the sensor capacitors with the parallel-plate capacitor (PPC) subcircuit, shown in Figure 5.7. External node voltages supply the plate position to the subcircuit. Other node voltages are used to output the force and torque produced by the capacitor. Static parameters include the location $(x_{co}, y_{co})$ and size $(L_{cx}, L_{cy})$ of the capacitor, the size of the fabricated air gap, $z_{co}$, and the etch-hole form factor, $\alpha_c$. Upper and lower actuators are distinguished by the sign of the fabricated air-gap value; capacitors above the plate are supplied with a negative fabricated air-gap value in the simulation. With this sign convention, a positive plate displacement reduces the size of the upper-actuator air gap.

Force, torque, and capacitance are calculated in the subcircuit as

$$F_{cz} = -\text{sgn}(z_{co}) \frac{\alpha_c \varepsilon L_{cx} L_{cy}}{(z_{co} + \Delta z + y_{co} \theta - x_{co} \phi)^2} V^2$$

\(^2\)The quantity $C_s V_c$ in Equation (5.22) is in units of farads, because the nonlinear voltage source, $B_{in}$, converts the product $V_c V$ to a voltage.
\[
\tau_{i\theta,i} = y_{co} F_{cz} \\
\tau_{i\phi,i} = -x_{co} F_{cz} \\
C = \frac{\alpha c L_{co} L_{cy}}{|z_{co} + \Delta z + y_{co} \theta - x_{co} \phi|}
\]

### 5.3.4 Testbed Simulation

A block diagram of the testbed SPICE simulation\(^3\) is given in Figure 5.8. On the left side of the diagram, voltage sources that represent electrostatic forces and torques are summed. The total force and torques are connected to the equivalent LCR-circuit model for each respective mode. Plate displacement and rotation are output as voltages from the LCR circuits and supplied as inputs to the parallel-plate capacitor subcircuit instances. The sensor and actuator time-varying-capacitor connections are shown on the right side of Figure 5.8. Both the capacitive divider and the unity-gain buffer circuit for each quadrant are modeled explicitly. The demodulator and comparator are represented by simple behavioral models.

A closed-loop step response of the testbed operating in air at a 50 kHz sampling rate is shown in Figure 5.9. At t=200 \(\mu s\), the position-reference inputs are stepped from 0 V to 56 mV, corresponding to a vertical plate displacement of about 0.5 \(\mu m\). Since all four position references are set to equal values, only the vertical (\(\Delta z\)) plate mode is excited. The plate rises to its new steady-state position in about 0.6 ms, with a constant velocity due to the overdamped mechanical system. The steady-state displacement exhibits sawtooth-shaped limit cycles at half the sampling frequency; the \(\Sigma-\Delta\) feedback force generates the limit cycles. One of the capacitive-divider output voltage waveforms is shown in Figure 5.9(c). The buffer’s voltage offset does not affect operation of the loop, because the position signal is modulated at 50 kHz. Most of this offset is an artifact of the simulator, which initially places extra charge on the high-impedance node of the capacitive divider. We have ignored the offset, but it may be eliminated in the future by proper setting of initial conditions and simulation parameters. As the displacement increases, the modulation amplitude also increases, but not symmetrically about the dc level. This asymmetry is caused by the nonlinear I-V characteristic of the diode that is connected between the high-impedance node and ground\(^4\). Demodulation of the sensor output voltage pro-

\(^3\)The HSPICE listing is in appendix B.

\(^4\)The diode sets the dc bias of the high-impedance node. See section 3.6 for more information.
Figure 5.8: Block diagram of the testbed, modeled in SPICE.
Figure 5.9: SPICE simulation of the testbed closed-loop step response in air. (a) Position reference input voltage, $V_{\text{ref}}$. (b) Plate displacement, $\Delta z$. (c) Modulated sensor output voltage (output from the upper-right capacitive divider), $V_{\text{sense}}$. (d) Peak-to-peak (demodulated) sensor voltage, $V_{\text{pp}}$. (e) Total vertical force acting on the plate, $F_z$. 
duces the peak-to-peak sensor voltage, $V_{pp}$, shown in Figure 5.9(d). This voltage tracks the position-reference input voltage. The total force acting on the the plate (Figure 5.9(e)) is, to first-order, quantized to two states; however, parallel-plate nonlinearity, force from the sensor electrodes, and the modulation voltage on the plate cause the total force to deviate from ideality. Changes in the actuator force levels arise from the change in parallel-plate air gap as the plate rises. The upper-actuator force level increases, while the lower-actuator force level is reduced. The $\pm 0.3$ V modulation voltage impressed on the plate produces an actuator force component at the sampling frequency, and also generates a dc force from the sensor electrodes.

Figure 5.10 shows simulation results of the testbed operation under vacuum, with a mechanical Q of 50. The plate oscillations increase to the mechanical limit stops (at $\pm 2 \mu m$) since there is no loop compensation to stabilize the system. The amplitude-modulated sensor output, and demodulated peak-to-peak sensor voltage are plotted in Figure 5.10(b-c). Large effects from the parallel-plate nonlinearity in the feedback force are evident in Figure 5.10(d).

With the preceding examples, we have demonstrated detailed SPICE simulation of the testbed using modular subcircuit models. Second-order effects in the system, such as the effects of modulation voltage on the force, are predicted by using this simulation approach. Further modeling improvements can be made without affecting the overall simulation.

One important drawback of SPICE is that the time to complete a simulation is usually very long; a 1 ms simulation, corresponding to 50 cycles at a 50 kHz sampling rate, takes about 30 minutes of CPU time on a DecStation 5100 with 32 Megabytes of RAM. Behavioral simulation, discussed next, alleviates the lengthy simulation times and allows a quick turnaround of analyses.

### 5.4 MATLAB Simulation

We have implemented a behavioral simulation of the testbed using the Simulink [94] extension to MATLAB. Simulink provides a convenient graphical user interface (GUI) for system simulation. A block diagram of the testbed simulation, taken from the GUI screen, is shown in Figure 5.11. Mechanical equations of motion and electrostatic actuator forces are modeled by their explicit transfer functions. The position sensor and buffer circuit are bundled together and modeled by Equation (3.178), which is repeated here for
Figure 5.10: SPICE simulation of the testbed under vacuum, with a mechanical Q of 50. (a) Plate displacement, $\Delta z$. (b) Modulated sensor output voltage (output from the upper-right capacitive divider), $V_{\text{sense}}$. (c) Peak-to-peak (demodulated) sensor voltage, $V_{pp}$. (d) Total vertical force acting on the plate, $F_z$. 
Figure 5.11: Block diagram of the testbed, modeled in MATLAB.
convenience.

\[ V_s = G_0 V_m \left( \frac{C_r - C_{so}}{C_r + C_{so} + C_p^i} \right) \left[ \frac{1 + \left( \frac{C_r - C_{sf}}{C_r - C_{so}} \right) \frac{\Delta z}{z_o}}{1 + \left( \frac{C_r + C_{sf} + C_p^o}{C_r + C_{so} + C_p^o} \right) \frac{\Delta z}{z_o}} \right] \]

Capacitive circuit models of the sensors and actuators are not included, and any electrical interactions among the components are ignored. SPICE simulation is more appropriate for examining these second-order electrical effects. Simulation examples are deferred to chapter 6, where we present experimental and simulated results for the testbed closed-loop step response in air and underdamped response at low pressure.
5.5 Limit-Cycle Analysis

The presence of nonlinear elements in a feedback loop can produce stable bounded oscillations, called limit cycles. In the testbed $\Sigma$-$\Delta$ loops, the nonlinearity of the one-bit quantizer gives rise to limit cycles. Describing-function analysis [95] is used to predict the existence of limit cycles, and the amplitude and frequency of the oscillation. A describing function for a nonlinear element is found by determining the fundamental frequency component of the response to a sinusoidal input signal. A low-pass transfer function is assumed to exist in the loop to attenuate higher-order harmonic components. In micromechanical digital-feedback systems, the high-frequency harmonics are attenuated by the two-pole rolloff of the mechanical mass-spring-damper.

In this section, two methods for evaluating limit cycles of the testbed will be discussed. The first method requires the derivation of a describing function for the entire feedback path. The second method requires the describing function of only the quantizer, with digital compensation in the feedback path being modeled separately as a $z$-transform. Both methods provide equivalent expressions for the loop transfer function. The limit-cycle condition occurs when the loop transfer function equals $-1$. Magnitude and frequency of the limit cycle are found from this condition.

Delays in the loop due to discrete-time sampling will vary because the limit-cycle frequency will never exactly equal a multiple of the sampling rate. The delay is also affected by circuit noise. If the noise is smaller than the plate oscillation amplitude, the assumption of a steady-state limit cycle can provide bounds on the oscillation amplitude and frequency. The ensuing analysis assumes that the delay is a fixed parameter and the limit-cycle frequency is constant.

5.5.1 Describing Function of the Feedback Path

A simple, single-axis model of the testbed, shown in Figure 5.12, is used to examine limit cycles of the vertical-displacement mode. The mass-spring-damper transfer function for the $\Delta z$ mode was presented in Equation (5.1), and is repeated here for convenience.

$$ H_z(s) = \frac{1}{m(s^2 + 2\zeta\omega_z s + \omega_z^2)} $$

The plate displacement is assumed to be sinusoidally varying with time.

$$ \Delta z = z_1 \sin(\omega t) \quad (5.27) $$
where $z_1$ is the displacement amplitude and $\omega$ is the frequency of the limit cycle. A single describing function, $H_{fb}(z_1, \omega)$, models the entire feedback path, including the position sensor, quantizer, compensation, and level shift.

A description of the feedback force is necessary to determine the describing function. Limit-cycle waveforms of linearized feedback force, $F_{fb}$, and plate displacement, $\Delta z$, are given in Figure 5.13 for an underdamped mechanical system with no loop compensation. After a reference crossing of the plate is sensed, a restoring force of magnitude $F_o$ is output to the appropriate actuator. The feedback response must occur at a clock edge, and will be delayed by $T_d$ from the time of the reference-crossing. A constant magnitude of actuator force, $F_o$, will be assumed in the analysis of this section and section 5.5.2. The value $F_o$ is referred to as the full-scale force, because external forces exceeding this value will pull the device out of closed-loop operation. The full-scale force must be large enough
Figure 5.14: Limit-cycle waveforms of linearized feedback force with compensation (solid line), and vertical plate displacement (dashed line).

Compensation is necessary for closed-loop stability when the mechanical system is underdamped. Without compensation and neglecting the extra delay $T_d$, the feedback force is exactly 180° out of phase with the displacement. Including the delay, the feedback force is more than 180° out of phase with the displacement, resulting in an unstable system. Introduction of the compensation term, $1 - z^{-1}$, in the feedback path, adds phase lead to the loop, and acts to stabilize the system. Limit-cycle waveforms with this type of lead compensation are given in Figure 5.14. After a reference crossing of the plate is sensed, a restoring force pulse of magnitude $F_v$, is output to the actuators. The lead pulse lasts one sample period, $T_s$, after which a smaller force magnitude, $F_o$, is applied until the plate crosses the displacement reference. The phase delay, $T_d$, arising from the position sampling in the loop, partly negates the effect of the compensation. The limit-cycle oscillation condition occurs when the phase lead from the compensation balances the phase delay due to the sampling. No compensation is a special case, specified when $F_v = F_o$.

The describing function, $H_{fb}(z_1, \omega)$, is the ratio of the fundamental frequency component of the feedback force to a sinusoidal plate displacement. The fundamental of the feedback force is found using Fourier analysis. One period of the feedback-force limit
cycle is approximated by a piecewise-linear expression.

\[
F_{fb} = \begin{cases} 
+ F_v & ; -\frac{T}{2} + T_d < t \leq -\frac{T}{2} + T_s + T_d \\
+ F_o & ; -\frac{T}{2} + T_s + T_d < t \leq T_d \\
- F_v & ; T_d < t \leq T_s + T_d \\
- F_o & ; T_s + T_d < t \leq \frac{T}{2} + T_d 
\end{cases} \quad (5.28)
\]

where \( T \) is the period of the limit cycle, and the plate displacement is given by Equation (5.27).

The feedback force can be expressed as a sum of complex exponential basis functions.

\[
F_{fb} = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega t} \quad (5.29)
\]

where \( \omega = \frac{2\pi}{T} \) is the limit cycle frequency. The Fourier coefficients, \( A_n \), are determined by first multiplying Equation (5.28) by \( e^{-jn\omega t} \) and integrating over a period.

\[
\int_{-T/2}^{T/2} F_{fb} e^{-jn\omega t} dt = \sum_{n=0}^{\infty} \int_{-T/2}^{T/2} A_n e^{j(n-m)\omega t} dt \quad (5.30)
\]

Since the exponential basis functions are orthogonal, all terms in the summation are zero except when \( n = m \). Equations (5.28) and (5.30) are combined and solved for the coefficients.

\[
A_m = -\frac{1 - (-1)^m}{2jm\pi} e^{-jm\omega T_d} \left[ F_v + F_o - (F_v - F_o) e^{-jm\omega T_s} \right] \quad (5.31)
\]

The fundamental component of feedback force, \( F_{fb1} \), is the sum of the \( |m| = 1 \) contributions to Equation (5.29).

\[
F_{fb1} = A_1 e^{jt\omega} + A_{-1} e^{-j\omega t} = -\frac{2}{\pi} e^{-j\omega T_d} \left\{ (F_v + F_o) \sin(\omega t) - (F_v - F_o) \sin[\omega(t - T_s)] \right\} \quad (5.32)
\]

A feedback describing function, \( H_{fb}(z_1, \omega) \), is determined by finding the ratio of the amplitude and offset in phase of Equations (5.32) and (5.27).

\[
H_{fb}(z_1, \omega) = -\frac{2}{\pi z_1} e^{-j\omega T_d} \left[ F_v + F_o - (F_v - F_o) e^{-j\omega T_s} \right] \quad (5.33)
\]

Feedback nonlinearity with position can be included in the Fourier analysis, and will be introduced in section 5.5.4. Determination of the loop transmission and limit-cycle condition is deferred to the next section.
5.5.2 Describing Function of Only the Quantizer

In the preceding section, the describing function for the entire feedback path was derived. As an alternative, the quantizer and digital feedback can be modeled separately, as illustrated in Figure 5.15. The position sensor, demodulator, and comparator are modeled together in the quantizer describing function $H_q(z_1, \omega)$. The quantizer has an output value of $+1$ or $-1$, corresponding to the plate being above or below the position reference.

The quantizer describing function can be found using the Fourier analysis method of the preceding section. In Figure 5.13, we recognize that the limit cycle output from the quantizer is a square wave delayed by $T_d$ from the displacement. The first-harmonic amplitude of the square wave is $4/\pi$, which is divided by the sinusoidal displacement amplitude to determine the describing-function gain.

$$H_q(z_1, \omega) = \frac{4}{\pi z_1} e^{-j\omega T_d} \quad (5.34)$$

Digital compensation and level shift are expressed as a z-transform, $H_{fb}(z)$.

$$H_{fb}(z) = -F_v \left[ 1 + G_v \frac{(1 - z^{-1})}{2} \right] \quad (5.35)$$

where $G_v = F_v/F_o - 1$ is the adjustable lead-compensation gain. Since a unit delay is the sampling period, $T_s$, we can substitute $z = e^{j\omega T_s}$ in Equation (5.35) and form the continuous-time transfer function, $H_{fb}(e^{j\omega T_s})$. The product $H_q(z_1, \omega) H_{fb}(e^{j\omega T_s})$ is equivalent to the feedback describing function derived in the preceding section, given by Equation (5.33).

The system loop gain, $L(z_1, \omega)$, is the product of Equations (5.1), (5.35), and (5.34).

$$L(z_1, \omega) = -H_z(j\omega) H_q(z_1, \omega) H_{fb}(e^{j\omega T_s}) \frac{4 F_v e^{-j\omega T_d} \left[ 1 + G_v (1 - e^{-j\omega T_s})/2 \right]}{\pi z_1 m (-\omega^2 + j2\zeta \omega \omega + \omega^2)} \quad (5.36)$$
The amplitude, $z_1$, and frequency, $\omega$, of the limit cycle are found by setting the loop gain equal to $-1$. The two transcendental equations, $\Re \{ L(z_1, \omega) \} = -1$ and $\Im \{ L(z_1, \omega) \} = 0$, can be solved iteratively for $z_1$ and $\omega$.

\[
\Re \{ L(z_1, \omega) \} = \frac{4F_o \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \cos(\omega T_d) - (G_v/2) \cos[\omega(T_s + T_d)] \right)}{2\zeta \omega \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \sin(\omega T_d) - (G_v/2) \sin[\omega(T_s + T_d)] \right)} - \frac{\pi z_1 \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \sin(\omega T_d) - (G_v/2) \sin[\omega(T_s + T_d)] \right)}{\left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \cos(\omega T_d) - (G_v/2) \cos[\omega(T_s + T_d)] \right)}
\]

\[
\Im \{ L(z_1, \omega) \} = \frac{4F_o \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \cos(\omega T_d) - (G_v/2) \cos[\omega(T_s + T_d)] \right)}{2\zeta \omega \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \sin(\omega T_d) - (G_v/2) \sin[\omega(T_s + T_d)] \right)} - \frac{\pi z_1 \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \sin(\omega T_d) - (G_v/2) \sin[\omega(T_s + T_d)] \right)}{\left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \cos(\omega T_d) - (G_v/2) \cos[\omega(T_s + T_d)] \right)}
\]

(5.37)

(5.38)

For the high-$Q$ case ($\zeta \approx 0$), Equations (5.37) and (5.38) are simplified to two simultaneous equations.

\[
\left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \sin(\omega T_d) - (G_v/2) \sin[\omega(T_s + T_d)] = 0
\]

(5.39)

\[
z_1 = \frac{4F_o \left( \left( 1 + G_v/2 \right) \cos(\omega T_d) - (G_v/2) \cos[\omega(T_s + T_d)] \right)}{\pi \omega \left( \left( \omega^2 - \omega^2 \right) \left( 1 + G_v/2 \right) \sin(\omega T_d) - (G_v/2) \sin[\omega(T_s + T_d)] \right)}
\]

(5.40)

The limit-cycle frequency is given implicitly by Equation (5.39). During a period of the limit cycle, at least one sample period must be spent pulling up on the plate and one spent pulling down. Several solutions for $\omega$ are mathematically possible, but at most one solution satisfies the constraint $T \geq 2T_s$. The amplitude, $z_1$, is found by substituting the value for $\omega$ in Equation (5.40).

For the undamped case, a minimum compensation gain, $G_{v,\text{min}}$, exists which gives a bounded displacement.

\[
G_{v,\text{min}} = \frac{2}{\sin[\omega z(T_s + T_d)] / \sin(\omega T_d) - 1}
\]

(5.41)

For $G_v < G_{v,\text{min}}$, the plate oscillates at the resonant frequency with increasing amplitude, eventually hitting the limit stops. If the sampling rate is much larger than the resonant frequency, $G_{v,\text{min}} \approx 2T_d/T_s$. Any sampled-data system will have a maximum delay equal to at least the sampling period, so use of $2 - z^{-1}$ compensation ($G_v = 2$) will not stabilize the system. Figure 5.16 is a plot of $G_{v,\text{min}}$ versus the ratio of sampling frequency to resonant frequency, with the sampling delay set to $1.5T_s$. The $f_s/f_z$ ratio must be greater than about 10 to stabilize the system using a practical compensation gain value. For the testbed measurements, $f_s/f_z = 21.1$, $G_{v,\text{min}} = 3.52$, and $G_v = 4$. Larger values of compensation
gain require a larger actuator size or higher driving voltage. Fabrication constraints limit the size of the actuators. Large voltages require special circuitry and increase switching noise.

We find the sampling delay for the testbed system by examining the implementation of the position-sensing circuit. The position sensor voltage is demodulated by subtracting the values sampled at times $T_s/3$ and $5T_s/6$ after the rising clock edge. The comparator output, $y(t)$, is given by

$$y(t) = \text{sgn} \left[ V_{\text{sense}} \left( nT_s - \frac{2T_s}{3} \right) - V_{\text{sense}} \left( nT_s - \frac{T_s}{6} \right) \right]$$

(5.42)

where $n$ is an integer specifying the clock cycle. Assuming the sensor voltage is linearly varying with time between samples, the displacement is observed at approximately $5T_s/12$ before the comparator switches. If we also neglect the nonlinearity of the position sensor, a simple relation between the comparator output and displacement results.

$$y(t) = \text{sgn} \left[ \Delta z \left( nT_s - \frac{5T_s}{12} \right) \right]$$

(5.43)

The displacement can cross the reference value just before or right after the observation,
Figure 5.17: Limit cycle amplitude and frequency versus lead-compensation gain, $G_v$, with $T_d = 5/12 T_s, T_s, 17/12 T_s$. $f_s = 100$ kHz, $F_o = 116$ nN, $f_z = 4.71$ kHz, $Q = 50000$. (a) Amplitude, $z_1$. (b) Frequency, $\omega$.

providing bounds on the delay, $T_d$.

$$\frac{5}{12} T_s \leq T_d \leq \frac{17}{12} T_s$$  \hspace{1cm} (5.44)

Noise at the comparator input can increase the bounds, but will be neglected. Using these extreme values of $T_d$, maximum and minimum bounds for limit-cycle frequency and amplitude can be calculated.

The displacement limit-cycle amplitude and frequency are plotted versus lead-compensation gain in Figure 5.17. Parameters are set to emulate the testbed operating in a vacuum with a sampling frequency of 100 kHz. Other system parameter values are$^5$. $F_o =$

$^5$We assume a nominal polysilicon film thickness of 2 $\mu$m in calculating these parameters, so the resonant
116 nN, $f_z = 4.71$ kHz, and $Q = 50000$. Curves for three different sampling delay values are shown: minimum delay ($5T_s/12$), maximum delay ($17T_s/12$), and an intermediate delay ($T_s$). The curves where $T_d$ is set to the maximum and minimum delay provide bounds on the amplitude and frequency of the limit cycle. For compensation values below $G_{v, \text{min}}$, the plate oscillates at the resonant frequency, $f_z$, until it hits the limit stops. The amplitude rapidly decreases when $G_v$ is increased above $G_{v, \text{min}}$. Compensation gain values above 5 keep the limit-cycle amplitude between 59 nm and 311 nm. An optimal value for the lead-compensation gain is around $G_v = 6.4$, which minimizes the upper bound on amplitude. As $G_v$ is increased further, the frequency approaches its maximum value of $\omega = 1/\pi/(2T_d + T_s)$. The value of limit-cycle frequency can not reach the Nyquist frequency, $f_s/2$, because of the non-zero sampling delay.

Limit-cycle amplitude and frequency are plotted in Figure 5.18 with the sampling rate increased to 1 MHz. The limit-cycle frequency values are ten times larger, and the amplitude about 100 times smaller than the values in Figure 5.17. However, lead compensation gain above about 4 is still necessary to ensure a bounded limit cycle. The optimum value for $G_v$ is now 5.6, its value decreasing slightly with increasing sampling rate.

Effects of sampling frequency and damping on the limit cycle are illustrated in Figure 5.19. When the system is mechanically underdamped, the change in amplitude with sampling rate is $-40 \text{ dB/decade}$. An overdamped system exhibits a $-20 \text{ dB/decade}$ change of amplitude with $f_s$, as shown by the $\zeta_z = 100$ case in Figure 5.19(a). At low sampling rates, the amplitude decreases linearly with increasing damping. The amplitude is insensitive to the damping factor at high sampling rates. Limit-cycle frequency increases linearly with sampling rate, except in a transition region where the system is near critical damping. The sampling rate and damping factor affect the point at which the transition from a dominant one-pole limit-cycle response to a two-pole response occurs. The dotted line on Figure 5.19(b) separates the two regions. The displacement waveform approaches a sawtooth pattern with frequency $f_s/4$ when a large damping factor is combined with a low sampling rate.

### 5.5.3 Non-Zero External Force

frequency value is somewhat different from the value of 3.7 kHz stated in chapter chapter results.
Figure 5.18: Limit cycle amplitude and frequency versus lead-compensation gain, $G_v$, with $T_d = \frac{17}{12} T_s$, $T_s$, $\frac{5}{12} T_s$, $f_s = 1$ MHz, $F_o = 116$ nN, $f_z = 4.71$ kHz, $Q = 50000$. (a) Amplitude, $z_1$. (b) Frequency, $\omega$. 
Figure 5.19: Limit cycle amplitude and frequency versus sampling frequency, $f_s$, with various damping factors. $T_d = 17/12 T_s$, $G_v = 6$, $F_o = 116$ nN, $f_z = 4.71$ kHz. (a) Amplitude, $z_1$. (b) Frequency, $\omega$. 
Figure 5.20: Limit-cycle waveforms of linearized feedback force with compensation (solid line), and vertical plate displacement (dashed line). The external force is balanced by feedback with duty cycle, $D$.

The analysis in sections 5.5.1 and 5.5.2 assume that the external force was zero. Gravity, accelerating fields, the position sensors, and non-zero position reference signals produce forces acting on the plate, which the feedback force must balance. In this section, we will investigate the effect of external and offset forces on the limit cycle.

The first step is to determine the fundamental components of the displacement and quantizer output. If a steady-state limit cycle exists, the duty cycle of the digital feedback is adjusted so the sum of forces acting on the plate is zero. Feedback with duty cycle, $D$, is shown in Figure 5.20. The displacement limit cycle is offset by $z_{os}$ and delayed $T_z$ from the position reference crossing.

$$\Delta z = z_{os} + z_1 \sin \left[ \omega (t - T_z) \right]$$  \hspace{1cm} (5.45)

where $z_1$ is the limit cycle amplitude. The time reference is set such that $\Delta z(t = 0) = 0$ and $\Delta z(t = DT) = 0$. By solving these equations, the offset and delay are found.

$$z_{os} = -\frac{z_1 \sin(2 \pi D)}{\sqrt{2 \left[ 1 - \cos(2 \pi D) \right]}}$$  \hspace{1cm} (5.46)

$$T_z = \frac{1}{\omega} \tan^{-1} \left[ \frac{\sin(2 \pi D)}{1 - \cos(2 \pi D)} \right]$$  \hspace{1cm} (5.47)

Equations (5.46) and (5.47) are substituted into (5.45), giving the displacement in terms of the duty cycle.

$$\Delta z = \frac{z_1 \left[ -\sin(2 \pi D) + \sin(\omega t) - \sin(\omega t - 2 \pi D) \right]}{\sqrt{2 \left[ 1 - \cos(2 \pi D) \right]}}$$  \hspace{1cm} (5.48)

The duty cycle is found by equating the average feedback force and the external
As an intermediate result of this analysis, we can estimate the dc loop gain. First, we recognize that the closed-loop gain equals the displacement offset divided by the external force.

\[
\frac{z_{oz}}{F_{ext}} = \frac{H_z(0)}{1 + L(z_1, 0)}
\]

We arrive at the expression for dc loop gain by combining Equations (5.50), (5.46) and (5.49) and rearranging terms.

\[
L(z_1, 0) = \frac{2F_{ext} \cos(0.5\pi F_{ext}/F_o)}{k_z z_1 \sin(\pi F_{ext}/F_o)} + 1
\]

where \(k_z\) is the spring constant for vertical displacement. The loop gain value is a minimum when \(F_{ext} = 0\) and maximum when \(|F_{ext}| = F_o\).

\[
\frac{2F_o}{\pi k_z z_1} + 1 \leq L(z_1, 0) \leq \frac{F_o}{k_z z_1} + 1
\]

Nominal parameter values for the testbed in air are \(F_o = 0.12\ \mu N\), \(k_z = 0.42\ \text{N/m}\), and \(z_1 = 25\ \text{nm}\). Figure 5.21 is a plot of loop gain of the testbed versus normalized external force. The loop gain value lies between 8.1 and 12.1, depending on the external force. A much higher loop gain is desirable to reduce nonlinear effects in microsensor and microcontroller applications. The loop gain increases with decreasing spring constant and increasing sampling frequency. The effect of dc loop gain on acceleration signal to noise ratio will be discussed in section 5.6.2.
Now, we turn our attention to determining the amplitude and phase of the quantizer output. The first harmonic output from the quantizer, \( y_1(t) \), is found using the Fourier analysis method of section 5.5.1.

\[
y_1(t) = -\frac{2}{\pi} e^{-j\omega T_d} \left[ \sin(\omega t) - \sin(\omega t - 2\pi D) \right]
\]  
(5.53)

When \( D = 0.5 \), this equation simplifies to the \( F_{\text{ext}} = 0 \) case, given by Equation (5.32). Inspecting the phase of Equations (5.48) and (5.53), we see that the delay of the first harmonic of the displacement and quantizer output differ by \( T_d \), independent of duty cycle.

The quantizer describing function consists of a gain term with delay \( T_d \).

\[
H_{v}(z_1, \omega) = \frac{2\sqrt{2}[1 - \cos(2\pi D)]}{\pi z_1} e^{-j\omega T_d} = \frac{4}{\pi z_1} \cos \left( \frac{\pi F_{\text{ext}}}{2F_o} \right) e^{-j\omega T_d}
\]  
(5.54)

If we increase the magnitude of the external force, the fundamental component of the feedback force decreases in amplitude. The amplitude is a maximum at a 50% duty cycle, where \( F_{\text{ext}} = 0 \). The phase of the quantizer describing function is not affected by external force, so we can still use Equation (5.39) to calculate the limit-cycle frequency (assuming high-\( Q \) operation). We include the effects of external force on limit-cycle amplitude by reducing the right side of Equation (5.40) by a factor of \( \cos(0.5\pi F_{\text{ext}}/F_o) \).

\[
z_1 = \frac{4F_o \cos(0.5\pi F_{\text{ext}}/F_o)}{\pi m} \left\{ \frac{(1 + G_u/2) \cos(\omega T_d) - (G_u/2) \cos[\omega(T_s + T_d)]}{\omega^2 - \omega_s^2} \right\}
\]  
(5.55)

In general, we desire to reduce the limit-cycle amplitude in order to minimize nonlinear effects of position. The largest value of limit-cycle amplitude occurs for zero external force, so the effect of external force can be neglected in a worst-case analysis.

### 5.5.4 Feedback Nonlinearity With Position

Figure 5.22 illustrates the effect of position nonlinearity on the feedback force. The force is applied by parallel-plate actuators, which have an inverse-square dependence on the plate spacing.

\[
F_{\text{fp}}(t, \Delta z) = H_{\text{pp}}(\Delta z) F_{\text{lin}}(t) = \begin{cases} 
+ F_{\text{lin}}(t)/(1 - \Delta z/z_o)^2 & \text{; upper actuator} \\
- F_{\text{lin}}(t)/(1 + \Delta z/z_o)^2 & \text{; lower actuator}
\end{cases}
\]  
(5.56)

where \( z_o \) is the reference spacing between the plates, \( F_{\text{lin}}(t) \) is the linearized force, and \( H_{\text{pp}}(\Delta z) \) is the parallel-plate nonlinearity term. The reference position is assumed to be centered between the upper and lower actuators. A first-order approximation for the parallel-plate nonlinearity term is valid when displacements are small compared to the reference
Figure 5.22: Limit-cycle waveforms of nonlinear feedback force with compensation (solid line), and vertical plate displacement (dashed line).

\[
H_{pp}(t) \approx \begin{cases} 
1 + (2z_1/z_o) \sin(\omega t) & ; \text{upper actuator} \\
1 - (2z_1/z_o) \sin(\omega t) & ; \text{lower actuator}
\end{cases}
\]  \hspace{1cm} (5.57)

where \( \Delta z \) is given by Equation (5.48) with the external force set to zero.

We can derive a describing function which models the quantizer and the actuator nonlinearity together. The compensation is introduced later as a multiplicative term in the expression for loop gain. The limit cycle generated when the quantizer output is passed through the parallel-plate nonlinearity is given by \( H_{pp}(t)g(t) \). Fourier coefficients of \( H_{pp}(t)g(t) \) are calculated using the analysis method from section 5.5.1.

\[
A_m = \frac{1}{T} \left\{ \left[ \int_{T_d}^{T/2+T_d} \left[ 1 - \frac{2z_1}{z_o} \sin(\omega t) \right] e^{-jm\omega t} dt \right] - \left[ \int_{T/2+T_d}^{T+T_d} \left[ 1 + \frac{2z_1}{z_o} \sin(\omega t) \right] e^{-jm\omega t} dt \right] \right\}
\]  \hspace{1cm} (5.58)

We compare the first-harmonic component with the sinusoidal displacement, providing us with a quantizer describing function which includes the parallel-plate nonlinearity.

\[
H_q(z_1, \omega) = \frac{4}{\pi z_1} e^{-j\omega T_d} - \frac{2}{z_o} \hspace{1cm} (5.59)
\]

The nonlinearity produces an additional first-harmonic component of the feedback which is in phase with the plate displacement. Limit-cycle amplitude and frequency are found by setting the loop gain to \(-1\), and solving the resulting simultaneous equations iteratively.

\[
L(z_1, \omega) = \frac{F_o \left[ (4/\pi/z_1)e^{-j\omega T_d} - 2/z_o \right] \left[ 1 + G_o(1 - e^{-j\omega T_s})/2 \right]}{m \left( -\omega^2 + j2\zeta\omega \omega + \omega^2 \right)} = -1 \hspace{1cm} (5.60)
\]
Figure 5.23: Limit cycle amplitude and frequency versus sampling frequency, $f_s$, with various damping factors. Actuator nonlinearity is included in the calculations. $T_d = \frac{17}{12} T_s$, $G_v = 6$, $F_o = 116$ nN, $f_z = 4.71$ kHz. (a) Amplitude, $z_1$. (b) Frequency, $\omega$.

Equations (5.61) and (5.62) are simplified expressions for the mechanically underdamped case ($\zeta_z \approx 0$).

\[
(\omega^2 - \omega_z^2) \left\{ \left( 1 + \frac{G_v}{2} \right) \sin(\omega T_d) - \frac{G_v}{2} \sin [\omega(T_s + T_d)] + \frac{\pi G_v z_1}{4 z_o} \sin(\omega T_s) \right\} = 0 \quad (5.61)
\]

\[
z_1 = \left[ \frac{4 F_o \{(1 + G_v/2) \cos(\omega T_d) - (G_v/2) \cos[\omega(T_s + T_d)]\}}{\pi m(\omega^2 - \omega_z^2)} \right] \times \left[ 1 + \frac{2 F_o \{ (1 + G_v/2) - (G_v/2) \cos(\omega T_s) \}}{z_o m(\omega^2 - \omega_z^2)} \right]^{-1} \quad (5.62)
\]

The effect of actuator nonlinearity on the limit-cycle amplitude is illustrated in Figure 5.23. Data represented by the solid lines includes the actuator nonlinearity in the calculations. The dashed lines are the linear curves plotted previously in Figure 5.19. In the underdamped cases, the sampling frequency must be much larger than the values predicted by linear theory to avoid instability. When the system is overdamped, including nonlinearity in the calculations modifies the results by no more than about 10%.
5.6 Noise Analysis

5.6.1 Brownian Noise

Brownian noise analysis is important in force sensors and accelerometers, where the noise force directly affects the sense signal. In contrast, equivalent Brownian noise position does not limit the performance of closed-loop micropositioners. Feedback nulls the position error, so the equivalent Brownian noise position is reduced from its open-loop value by the effective loop gain. In the testbed Σ-Δ loop, limit-cycles, caused by the high-frequency force-feedback pulses, are the primary source of chatter in the plate. Therefore, we will focus our attention on the equivalent Brownian noise acceleration of the testbed.

Brownian noise was discussed in section 3.6.3.3, where we found that it is often greater than circuit noise in a single-mode surface microsystem. In the multi-mode testbed, a Brownian noise source is associated with each vibrational mode. Noise from the $z$, $\theta$, and $\phi$ modes couples to the signal outputs that control each corner of the plate; the noise associated with higher modes can be neglected. Using solutions analogous to Equation (3.232), we find that the noise contributions from the $z$, $\theta$, and $\phi$ modes are

$$\overline{I}_{z}^2 = 4k_BT\left(2\zeta_z m\omega_z\right)\Delta f \quad (5.63)$$
$$\overline{\tau}_{\theta}^2 = 4k_BT\left(2\zeta_{\theta} I_{\theta}\omega_\theta\right)\Delta f = K_B\phi I_y^2 \overline{I}_z^2 \quad (5.64)$$
$$\overline{\tau}_{\phi}^2 = 4k_BT\left(2\zeta_{\phi} I_{\phi}\omega_{\phi}\right)\Delta f = K_B\phi I_x^2 \overline{I}_z^2 \quad (5.65)$$

where $\overline{\tau}_{\theta}$ and $\overline{\tau}_{\phi}$ are the Brownian noise torque about the $x$-axis and $y$-axis, respectively. We have made the assumption that damping coefficients of the rotational modes are given by Equations (3.34) and (3.35). Since the plate is perforated with many holes, the rotational damping will be comparable to the vertical damping, and cannot be neglected. Equivalent Brownian noise acceleration for the three modes is

$$\overline{a}_{eq, z}^2 = 4k_BT\left(\frac{2\zeta_z \omega_z}{m}\right)\Delta f \quad (5.66)$$
$$\overline{a}_{eq, \theta}^2 = 4k_BT\left(\frac{2\zeta_{\theta} \omega_{\theta}}{I_{\theta}}\right)\Delta f = \left(\frac{144 K_B\phi}{I_y^2}\right) \overline{a}_{eq, z}^2 \quad (5.67)$$
$$\overline{a}_{eq, \phi}^2 = 4k_BT\left(\frac{2\zeta_{\phi} \omega_{\phi}}{I_{\phi}}\right)\Delta f = \left(\frac{144 K_B\phi}{I_x^2}\right) \overline{a}_{eq, z}^2 \quad (5.68)$$

where $\overline{a}_{eq, \theta}^2$ and $\overline{a}_{eq, \phi}^2$ are the equivalent Brownian-noise angular acceleration.
The external modes of force and torque acting on the testbed are extracted by adding or subtracting the actuator force signals.

\[
\begin{bmatrix}
F_z \\
\tau_\theta \\
\tau_\phi
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
L_{ay} & L_{ay} & -L_{ay} & -L_{ay} \\
-L_{ax} & L_{ax} & L_{ax} & -L_{ax}
\end{bmatrix}
\begin{bmatrix}
F_{ur} \\
F_{ul} \\
F_{ll} \\
F_{lr}
\end{bmatrix}
\] (5.69)

where the actuators are located at \( |x| = L_{ax} \) and \( |y| = L_{ay} \) from the center of the plate. The noise from each mode referred to the actuators is correlated. When each mode is extracted from the actuator force signals, only the noise from that mode is present; the other modal noise contributions are cancelled. For example, if all four actuator forces are summed, the correlated noise from the rotational modes is cancelled, resulting in a total noise force of \( \overline{F_z} \). Similarly, the total noise torques are \( \overline{\tau_\theta} \) and \( \overline{\tau_\phi} \), when referred to \( \tau_\theta \) and \( \tau_\phi \), respectively.

We can substitute the testbed parameter values given in Table A.1 into Equations (5.66) through (5.68) to calculate the Brownian noise in air. The resulting testbed equivalent Brownian noise acceleration is 3.7 milli-G and the equivalent noise angular acceleration is approximately 190 Hz/s (684000/s²), assuming a 50 Hz signal bandwidth.

The noise force can be referred to the individual actuator inputs by defining the four actuator forces as functions of \( F_z \), \( \tau_\theta \), and \( \tau_\phi \). However, the testbed system is over-constrained, so we would need an extra equation to invert Equation (5.69). Instead, we will analyze the situation in our acceleration experiments\(^6\), where we have deactivated the lower-right actuator to eliminate the extra constraint. Then, the actuator forces, in terms of the modal forces and torques, are

\[
\begin{bmatrix}
F_{ur} \\
F_{ul} \\
F_{ll} \\
F_{lr}
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
1 & 0 & -1/L_{ax} \\
0 & 1/L_{ay} & 1/L_{ax} \\
1 & -1/L_{ay} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
F_z \\
\tau_\theta \\
\tau_\phi
\end{bmatrix}
\] (5.70)

and the equivalent actuator noise forces when only the three actuators are operating are

\[
\overline{F_{eq,ur}^2} = \frac{1}{2} \left( \overline{F_z^2} + \overline{\tau_\phi^2} \right) = \frac{\overline{F_z^2}}{2} \left[ 1 + K_{B\phi} \left( \frac{L_x}{L_{ax}} \right)^2 \right] \] (5.71)

\(^6\)See section 6.8 for the experimental results.
\[ \overline{f_{\text{eq,}\text{ul}}} = \frac{1}{2} \left( \frac{\sigma^2_{\theta}}{L_{\text{ay}}} + \frac{\sigma^2_{\phi}}{L_{\text{ax}}} \right) = \frac{\overline{f_z^2}}{2} \left[ K_{B\theta} \left( \frac{L_{\text{ay}}}{L_{\text{ay}}} \right)^2 + K_{B\phi} \left( \frac{L_{\text{ax}}}{L_{\text{ax}}} \right)^2 \right] \] (5.72)

\[ \overline{f_{\text{eq,}\text{ll}}} = \frac{1}{2} \left( \frac{\sigma^2_{\phi}}{L_{\text{ay}}} \right) = \frac{\overline{f_z^2}}{2} \left[ 1 + K_{B\phi} \left( \frac{L_{\text{ay}}}{L_{\text{ay}}} \right)^2 \right] \] (5.73)

Since Brownian noise from all three modes is balanced by the actuator feedback, noise on a single signal of a quadrant is larger than the noise on the sum of all four signals, \( \overline{f_z^2} \). In a 50 Hz bandwidth, the values of equivalent actuator noise acceleration are \( \sqrt{f_{\text{eq,ul}}^2} = 7.13 \) milli-G, \( \sqrt{f_{\text{eq,ll}}^2} = 9.13 \) milli-G, and \( \sqrt{f_{\text{eq,ul}}^2} = 6.80 \) milli-G; all are larger than \( \sqrt{f_z^2} \).

5.6.2 Sigma-Delta Quantization Noise

5.6.2.1 Sampled-Data Representation

We will use the single-loop representation in Figure 5.24(a) to analyze quantization noise in the testbed. A sampled-data version of the loop is shown in Figure 5.24(b). The micromechanical mass-spring-damper and the sampling switch are approximated by a second-order z-transform, \( H(z) \).

\[ H(z) = \frac{K_n z^{-1}}{1 - \alpha z^{-1} + \beta z^{-2}} \] (5.74)

where \( \alpha \) and \( \beta \) are fixed coefficients, determined from the second-order mechanical transfer function, and \( K_n \) is the loop gain normalization factor. Based on the limit-cycle analysis developed earlier, The high-gain nonlinear quantizer enforces a loop gain value of \(-1\) at the Nyquist rate, \( f_s/2 \). This loop-gain approximation is valid for overdamped systems and for underdamped systems that are compensated and sampled at a frequency much higher than the resonant frequency. The comparator in Figure 5.24(b) is modeled as an added source of white noise with a total mean square noise given by [90]

\[ \overline{\epsilon_{\text{TOT}}^2} = \frac{\Delta^2}{12} \] (5.75)

where \( \Delta \) is the difference in the two quantizer output levels. Quantization noise is limited in bandwidth to the Nyquist rate, so the noise power in a frequency interval \( \Delta f \) is

\[ \overline{\epsilon_f^2} = \frac{\overline{\epsilon_{\text{TOT}}^2}}{f_s/2} = \left( \frac{\Delta^2}{6 f_s} \right) \Delta f \] (5.76)
Figure 5.24: Single-loop representation of the testbed. (a) Continuous-time block diagram. (b) Sampled-data version of (a).
5.6.2.2 Equivalent Output-Reflected Quantization Noise

We derive the equivalent noise acceleration by first referring the quantization noise to the output, \( y[n] \) (see Figure 5.24(b)). The transfer function, \( Y(z)/E_q(z) \), of the noise input to the output is

\[
\frac{Y(z)}{E_q(z)} = \frac{1}{1 + H(z) H'_n(z)} \tag{5.77}
\]

where \( H'_n(z) \) is the normalized z-transform in the feedback path, equal to

\[
H'_n(z) = 1 + G_v \left( \frac{1 - z^{-1}}{2} \right) \tag{5.78}
\]

Combining Equations (5.74), (5.77), and (5.78), we get

\[
\frac{Y(z)}{E_q(z)} = \frac{1 - \alpha z^{-1} + \beta z^{-2}}{1 + C z^{-1} + D z^{-2}} \tag{5.79}
\]

where \( C \equiv K_n - \alpha + K_n G_v / 2 \), and \( D \equiv \beta - K_n G_v / 2 \).

We now use Equation (5.79), and make the substitution \( z = e^{i\omega T_s} \), to obtain the equivalent output-reflected noise, \( \overline{e_y^2} \):

\[
\overline{e_y^2} = \left| \frac{Y(e^{i\omega T_s})}{E_q(e^{i\omega T_s})} \right|^2 \overline{e_y^2} = \frac{1 - \alpha \cos(\omega T_s) + \beta \cos(2\omega T_s)}{1 + C \cos(\omega T_s) + D \cos(2\omega T_s)}^2 + \left| \frac{1 - \alpha \sin(\omega T_s) - \beta \sin(2\omega T_s)}{1 + C \sin(\omega T_s) + D \sin(2\omega T_s)} \right|^2 \overline{e_y^2} \tag{5.80}
\]

If the signal bandwidth is much less than the sampling frequency \( (\omega T_s \ll 1) \), which would be the case for an oversampled converter, we can approximate Equation (5.80) by

\[
\overline{e_y^2} \approx \left( \frac{1 - \alpha + \beta}{1 + K_n - \alpha + \beta} \right)^2 \overline{e_y^2} \tag{5.81}
\]

At low frequencies, the equivalent output-reflected noise is independent of frequency. The lack of noise shaping is a consequence of the constant low-frequency gain of the mechanical mass-spring-damper. If the signal bandwidth extends beyond the mechanical transfer-function poles, then the approximation breaks down and the frequency terms in Equation (5.80) must be included in the noise calculation.

The coefficients, \( \alpha \) and \( \beta \), are found directly from the impulse response of the second-order mechanical system. The mechanical transfer function can be factored such that

\[
H(s) = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{1}{(s + a_1)(s + a_2)} \tag{5.82}
\]
where the roots, $-c_1$ and $-c_2$, are given by

$$c_1 = \zeta \omega_n - \frac{\omega_n}{\sqrt{\zeta^2 - 1}}$$
$$c_2 = \zeta \omega_n - \frac{\omega_n}{\sqrt{\zeta^2 - 1}}$$

For an overdamped system ($\zeta > 1$), the impulse response is

$$h(t)|_{\zeta > 1} = \frac{1}{2\omega_o} \left(e^{-a_1 t} - e^{-a_2 t}\right) u(t)$$

where $\omega_o \equiv \omega_n \sqrt{\zeta^2 - 1}$. The corresponding sampled-data impulse response is

$$h[n]|_{\zeta > 1} = \frac{1}{2\omega_o} (a_1^n - a_2^n) u[n]$$

where $a_1 = e^{-a_1 T_s}$ and $a_2 = e^{-a_2 T_s}$. We take the z-transform of Equation (5.86) and get

$$H(z)|_{\zeta > 1} = \left(\frac{a_1}{2\omega_o}\right) \frac{z^{-1}}{(1 - a_1 z^{-1})(1 - a_2 z^{-1})}$$

Finally, the coefficients in the denominator are matched to solve for $\alpha$ and $\beta$, yielding

$$a_1|_{\zeta > 1} = a_1 + a_2 = e^{-\zeta \omega_n T_s} \left(e^{\omega_n T_s} + e^{-\omega_n T_s}\right)$$
$$\beta|_{\zeta > 1} = a_1 a_2 = e^{-2\zeta \omega_n T_s}$$

For $\omega_o \ll f_s$ and moderate damping, $\alpha \approx 2$ and $\beta \approx 1$; in this case, the mechanical system behaves like an ideal sampled-data double integrator. A relatively low sampling rate or large damping produces a lossy single-pole mechanical response below the Nyquist frequency, where $\alpha \approx e^{-\omega_n T_s/(2\zeta)}$ and $\beta$ can be neglected.

An approximate value for the normalization factor can be determined by first finding the dc loop gain, $L(z)|_{z=1}$,

$$L(z)|_{z=1} = \left[\frac{-K_n \left(1 + 0.5 G_n(1 - z^{-1})\right) z^{-1}}{1 - \alpha z^{-1} + \beta z^{-2}}\right]_{z=1} = \frac{-K_n}{1 - \alpha + \beta}$$

For the overdamped case, we obtain

$$L(z)|_{z=1} \approx \frac{-K_n}{1 - e^{-c_1 T_s}} \approx \frac{-K_n}{c_1 T_s}$$
Since the loop transfer function has a pole at $\omega = c_1$, the gain at the Nyquist frequency will be $-1$ if $K_n = \pi$.

A similar analysis can be performed for the underdamped case ($\zeta < 1$); the resulting $\alpha$ and $\beta$ coefficients are

\[
\begin{align*}
\alpha|_{\zeta < 1} &= 2e^{-\zeta \omega_n T_s} \cos(\omega_n T_s) \\
\beta|_{\zeta < 1} &= \beta|_{\zeta > 1} = e^{-2\zeta \omega_n T_s}
\end{align*}
\]

and the normalization factor (for $\zeta \ll 1$) is $K_n = \pi^2$.

**5.6.2.3 Equivalent Quantization Noise Acceleration**

We find the equivalent noise acceleration by referring the equivalent output noise to the force input, and scale by $1/m$, and get

\[
\overline{a^2_{eq,q}} = \left( \frac{F_o}{m} \right)^2 \frac{2^2}{2^2} 
\]

Combining Equations (5.81), (5.88), (5.89) and (5.94), we obtain the equivalent input-referred noise acceleration in terms of the quantization noise. For an overdamped system,

\[
\overline{a^2_{eq,q}} \approx \left( \frac{F_o}{m} \right)^2 \left( \frac{\omega_n}{2\pi f_s} \right)^2 \frac{2^2}{2^2} 
\]

where we have used the oversampling approximation ($\omega T_s \ll 1$). The total noise acceleration is found by integrating Equation (5.95) over the signal frequency band, $f = 0$ to $f_o$, yielding

\[
\overline{a^2_{eq,q;TOT}} \approx \left( \frac{F_o}{m} \right)^2 \left( \frac{\Delta^2}{12} \right) \frac{2f_n^2 f_o}{\zeta^2 f_s^5} 
\]

where $f_n = \omega_n / 2\pi$. Doubling the sampling frequency results in a 9 dB noise reduction, as expected for a first-order $\Sigma$-$\Delta$ loop; however, reducing the signal bandwidth by half results in only a 3 dB noise reduction, because the noise spectral density is constant. Lowering the resonant frequency or increasing the damping will also reduce the input-referred noise. The damping dependence on quantization noise is reversed from that of the Brownian noise, for $\zeta > 1$.

Equivalent input-referred noise acceleration for a mechanically underdamped system is

\[
\overline{a^2_{eq,q}} \approx \left( \frac{F_o}{m} \right)^2 \left( \frac{\omega_n}{\pi f_s} \right)^4 \frac{2^2}{2^2} 
\]
Again, we assume $\omega T_s \ll 1$. Integrating the noise spectral density from $f=0$ to $f_o$, we obtain the total noise acceleration.

$$\overline{a_{n,TOT}} \mid_{\omega T_s \ll 1} \simeq \left( \frac{F_o}{m} \right)^2 \left( \frac{\Delta^2}{12} \right) \frac{f_o^4 f_o}{(f_s/2)^5} \tag{5.98}$$

The quantization noise acceleration is reduced by 15 dB/octave increase in the sampling rate, which matches the performance of the conventional second-order $\Sigma$-$\Delta$ loop. As in the overdamped case, the noise is reduced by 3 dB/octave decrease in the signal bandwidth. The noise is insensitive to the mechanical quality factor, but there is a noise reduction of 12 dB/octave due to a decrease in the resonant frequency.

The testbed mechanical parameters for operation in air are $m=0.47 \mu g$, $\zeta_z=12$, and $f_{n,z}=3.7$ kHz, corresponding to real poles at 150 Hz and 88 kHz. The non-dominant pole is above the sampling frequency of 50 kHz and does not affect the noise shaping. Assuming a full-scale feedback force of $\pm 293$ nN, ($\pm 64$ G) and a signal bandwidth of 50 Hz, and plugging these values into Equation (5.96), we calculate a value of 10.1 milli-G for total equivalent quantization noise acceleration.

### 5.6.3 Equivalent Noise Force from Interconnect Resistance

Another source of equivalent noise force arises from interconnect resistance, which adds thermal noise to the electrostatic actuator voltages. This noise is present on feedback actuators, sensor capacitors, and open-loop actuators (such as the testbed’s comb drives). Electrostatic force generated by these elements is proportional to the square of the applied voltage.

$$F = \gamma V_a^2 \tag{5.99}$$

where $\gamma$ is a proportionality constant with units of N/V$^2$. Equivalent noise acceleration due to an interconnect resistance, $R$, is

$$\overline{a_{R,eq}} = 4kT R \left( \frac{2\gamma V_a}{m} \right)^2 f_o \tag{5.100}$$

The total equivalent noise force from interconnect is found by summing the contributions from each actuator and sensor capacitor. For the testbed, the values for $\gamma$ and the applied voltage are:

$^5$These values assume that all four actuators are operating. In our acceleration measurements, we operate only three of the four actuators at the plate’s corners. Then, the full-scale acceleration is $\pm 48$ G and the total equivalent quantization noise acceleration is 7.5 milli-G.
upper actuator: \[ \gamma_u = 1.70 \text{ nN/V}^2 \]
\[ V_{a,u} = 7.15 \text{ V} \]
lower actuator: \[ \gamma_l = 2.45 \text{ nN/V}^2 \]
\[ V_{a,l} = 3.56 \text{ V} \]
sensor capacitor: \[ \gamma_s = 18.3 \text{ nN/V}^2 \]
\[ V_{a,s} = 0.3 \text{ V} \]

Using a worst-case resistance value of 35 kΩ and assuming a 50 % duty cycle between the upper and lower actuators, we calculate an equivalent noise acceleration of 0.05 milli-G in a 50 Hz bandwidth.

### 5.6.4 Total Noise Acceleration for the Testbed

Brownian noise \( \left( a_{\zeta,\text{TOT}}^2 \right) \), quantization noise \( \left( a_{q,\text{TOT}}^2 \right) \), electronic noise \( \left( a_{e,\text{TOT}}^2 \right) \) and interconnect noise \( \left( a_{R,\text{TOT}}^2 \right) \) contribute to the total equivalent noise acceleration of the testbed:

\[
\frac{a_{\text{TOT}}^2}{a_{\zeta,\text{TOT}}^2 + a_{q,\text{TOT}}^2 + a_{e,\text{TOT}}^2 + a_{R,\text{TOT}}^2} = \frac{1}{1.5}
\]

With the exception of electronic noise, values of these noise contributions have been calculated in the preceding sections. Electronic noise is primarily generated from the unity-gain buffer, discussed in sections 3.6.1.3 and 3.6.3.2. As shown in Figure 3.41, the diode shot noise current dominates the buffer noise at the 50 kHz sampling rate. Since the reference capacitors in our experiments are disconnected, the rms voltage signal into each buffer is around 0.1 V. Using Equation (3.195), we calculate a total noise voltage of 0.52 μV/√Hz, which is equivalent to a noise acceleration of 0.35 milli-G in a 50 Hz bandwidth. Noise from each buffer is uncorrelated, so the total noise contribution from the electronics is 0.7 milli-G.

We substitute the values for noise into Equation (5.101) and obtain the testbed equivalent noise acceleration in a 50 Hz bandwidth,

\[
\sqrt{a_{\text{TOT}}^2} = \left[ \left( 3.7 \text{ milli-G} \right)^2 + \left( 10.1 \text{ milli-G} \right)^2 + \left( 0.7 \text{ milli-G} \right)^2 + \left( 0.05 \text{ milli-G} \right)^2 \right]^{1/2}
\]

\[
= 10.8 \text{ milli-G}
\]

The total noise is dominated by quantization noise, with the Brownian noise being about 7 times less significant and the other noise contributions being negligible.