Optimizing Joint Erasure- and Error-Correction Coding for Wireless Packet Transmissions

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Motivation

- Often data has to be transmitted through a series of wireless and wired links.

- Performance bottleneck is then the wireless link:
  - Unreliable due to fading
  - Less bandwidth
  - Subject to interference

- Powerful and efficient coding possible across the whole data block.

- Individual packets are subject to fading when traveling across the wireless link:
  - Outage corrupts complete packet, otherwise negligible error rate
  - Not efficient to forward corrupted packets

- Use error-correction coding per packet -> view as an erasure channel.
Digital Fountain Codes

- Efficient erasure-correction codes now available
  - Digital Fountain principle - generate practically endless streams of encoded packets
  - Reception of a sufficient number of correct packets leads to high decoding probability
  - Small overhead (about 5% for reasonable size)

System Model – Assumptions

- End-to-end transport of a finite-size data block
  - Performance is dominated by a wireless link
  - Wireless link is well characterized by block fading model
    \[ y = hs + w \]

- Average signal-to-noise ratio on wireless link
  \[ \gamma = E[|s|^2] / |w|^2 \]

- Large feed-back delay
  - Usage of automatic repeat request (ARQ) not possible
Layered coding approach
- Erasure-correction coding across the data packets
- Error-correction coding per packet on the physical layer

Erasure-correction coding
- Block of $N_{data}$ bits is partitioned into $k$ packets
- Generate $K$ encoded packets with rate $r_n$

Error-correction coding
- Each packet of $N_s$ symbols carries $N_b$ bits
- Define coding rate as non-vanishing fraction of ergodic Capacity $C$

\[ N_{data} = kN_b \]
\[ r_n = \frac{k}{K} \]
\[ R_{phy} = \frac{N_b}{N_s} \]
\[ r_p = \frac{R_{phy}}{C} \]
System Model – Nakagami Model

- Capacity on Nakagami-$m$ block fading channel
  - Mutual information assuming Capacity achieving Gaussian codebooks
  - Ergodic capacity defined as average mutual information

\[ I = \log_2 \left( 1 + \gamma |h|^2 \right) \]

\[ C(\gamma, m) = E \left[ \log_2(1 + \gamma |h|^2) \right] = \log_2(e) e^{m/\gamma} \sum_{k=0}^{m-1} \left( \frac{m}{\gamma} \right)^k \Gamma \left( -k, \frac{m}{\gamma} \right) \]

- Correct physical layer decoding is achieved, if mutual information is above transmission rate

\[ p = \Pr (I < R_{\text{phy}}) \]
\[ = \Pr (|h|^2 < \alpha = \left( 2 R_{\text{phy}} - 1 \right) / \gamma) \]
\[ = 1 - \sum_{k=0}^{m-1} \frac{1}{k!} (m\alpha)^k e^{-m\alpha} \]
System Model – Performance

- Total outage probability of transmission
  - Depends on number of correctly received packets \( k' > k \)
  - Packet error detection based on CRC is perfect

\[
P_{\text{outage}} = \sum_{i=0}^{k'-1} \binom{K}{i} (1 - p)^i \cdot p^{K-i}
\]

- Define efficiency of data transfer

\[
\eta = \frac{N_{\text{data}}}{KN_sC} = \frac{k}{K} \cdot \frac{N_b}{N_sC} = r_n \cdot r_p
\]
Problem Statement

- Obvious trade-off necessary between $r_p$ and $r_n$
  - Smaller physical rate leads to less corrupted packets
  - Low network rate reduces vulnerability to packet loss

- Investigate two dual problems:
  1. Optimizing Performance under Resource Const.
     - Fix overall efficiency
     - Split resources between coding layers
     - Adhere to prescribed outage probability
     - Combine strengths of coding layers
Preliminaries

- For large $k'$ and $K$ approximate $P_{\text{outage}}$ as Gaussian

$$P_{\text{outage}} \approx Q\left(\frac{Kq - k'}{\sqrt{Kpq}}\right)$$

- Probability of correct transmission $q$

$$q = 1 - p$$

- Define a constant $\rho$

$$\rho = \frac{k'}{k}$$

- Portion of variable redundancy in $r_n$

$$\tilde{r}_n = \frac{k'}{K} = \rho r_n$$

- Simplify $P_{\text{outage}}$ using the definitions

$$P_{\text{outage}} = Q\left(\sqrt{\frac{\rho N}{N s C}} \frac{q - \tilde{r}_n}{\sqrt{\tilde{r}_n r_p pq}}\right)$$
Optimal Combining of Inter- and Intra-Packet Coding – Solution 1.

- Optimizing performance under resource constraint
  - Use equivalent obj. function
    \[ \max J(r_p, \tilde{r}_n) := \frac{q - \tilde{r}_n}{\sqrt{r_p \tilde{r}_n (1 - q) q}} \]
  - The Lagrange approach leads to:
    \[ \tilde{r}_n = \frac{-r_p q \dot{q}}{2q(1 - q) - r_p (2q - 1) \dot{q}} \]
    \[ \dot{q} = \frac{\partial q}{\partial r_p} \]
  - Solution is intersection with constraint

- Numerical Example
  - \( P_{\text{outage}} = 1 \) for \( r_p < 0.5 \)
  - Clear minimum for average SNR around \( r_p = 0.8 \)
    \[ \tilde{\eta}_0 = 0.5 \quad \frac{\rho N}{CN_s} = 2^8 \]
Minimizing efficiency under performance constraint
- Constraint and objective exchanged: dual problem
- Intersect with performance constraint instead

Numerical Example
- Plot looks concave with global maximum for all SNR
Optimal rates for Rayleigh and Nakagami-4 fading channel
Rayleigh PER is above $10^{-1}$ while Nakagami-4 is much lower
Consider an infinite data stream
- Outage probability goes to zero

\[
\lim_{N_{\text{data}} \to \infty} Q \left( \sqrt{\frac{\rho N_{\text{data}}}{N_s C}} \frac{q - \tilde{r}_n}{\sqrt{\tilde{r}_n r_p q (1 - q)}} \right) = 0
\]

- If erasure coding rate is below success \( r_n < q \)

This leads to a simpler optimization problem
- Outage problem is zero
- Erasure coding replaces lost packages

\[
\max \eta = \max_{r_p} r_p q
\]
Rate Optimization in a Special Case

Result I

On a Rayleigh fading channel, the physical rate maximizing $\eta = r_p q$ is given by:

$$r_p = \frac{W(\gamma)}{\ln(2)C}$$

With the Lambert-w function $W(\gamma)$

This leads to a network rate as:

$$\tilde{r}_n = \exp \left[ -\frac{1}{W(\gamma)} + \frac{1}{\gamma} \right]$$

At high SNR, we have:

$$\lim_{\gamma \to \infty} r_p = 1, \quad \lim_{\gamma \to \infty} \tilde{r}_n = 1$$

Using Results I and numerical optimization we plot $r_p$

Rayleigh shows distinctly different behavior for $m > 2$
For general Nakagami-$m$ fading channels, the optimal rates maximizing $\eta = r_p q$ at vanishing SNR are constant.

<table>
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<th>$m$</th>
<th>$r_p$</th>
<th>$r_n$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$e^{-1}\approx 0.368$</td>
</tr>
<tr>
<td>2</td>
<td>0.809</td>
<td>0.519</td>
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<tr>
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<tr>
<td>4</td>
<td>0.736</td>
<td>0.729</td>
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Layered coding approach leads to practical and efficient transmission scheme

A well-defined tradeoff exists, optimally allocating resources to both coding levels
- On severe fading channels, tendency is to use more erasure coding
- Investing in physical layer coding has worse payoff

For infinite data streams, closed form solutions show specific behavior for severe fading channels