# Derivation of Extended Kalman Filtering and Smoothing Equations

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**Abstract**

State estimation for nonlinear dynamical systems can be performed via local linearization of the nonlinearities. This Extended Kalman approach can be used for both filtering [1], [2] and smoothing. We follow the approach in [3] to derive the forward and backward Extended Kalman recursions. We assume that the reader is familiar with [3].

## 1 Introduction

We consider nonlinear dynamical systems of the following form:

$$\mathbf{x}_{t+1} = \mathbf{f}_t(\mathbf{x}_t) + \mathbf{w}_t \tag{1}$$

$$\mathbf{y}_t = \mathbf{g}_t(\mathbf{x}_t) + \mathbf{v}_t \tag{2}$$

where $\mathbf{x}_t \in \Re^k$ and $\mathbf{y}_t \in \Re^p$ are the state and output, respectively, at time $t$. The noise terms, $\mathbf{w}_t$ and $\mathbf{v}_t$, are zero-mean normally-distributed random variables with covariance matrices $Q_t$ and $R_t$, respectively. In general, the state update function $\mathbf{f}_t : \Re^k \to \Re^k$, the state-to-output mapping function $\mathbf{g}_t : \Re^k \to \Re^p$, and the covariance matrices of the noise variables can all vary with time. The initial state, $\mathbf{x}_1$, is normally-distributed with mean $\boldsymbol{\pi}_1$ and variance $V_1$.

As in [3], we assume that the parameters of the nonlinear dynamical system, namely $\mathbf{f}_t$, $\mathbf{g}_t$, $Q_t$, $R_t$, $\boldsymbol{\pi}_1$, and $V_1$ are known. Whereas the outputs are observed, the state and noise variables are hidden.

The Extended Kalman approach approximates the nonlinear system described by (1) and (2) with a linear system using first-order Taylor approximations

$$\mathbf{f}_t(\mathbf{x}_t) \approx \mathbf{f}_t(\mathbf{x}_t^t) + A_t(\mathbf{x}_t - \mathbf{x}_t^t) \tag{3}$$

$$\mathbf{g}_t(\mathbf{x}_t) \approx \mathbf{g}_t(\mathbf{x}_t^{t-1}) + C_t(\mathbf{x}_t - \mathbf{x}_t^{t-1}), \tag{4}$$

where

$$A_t = \left.\frac{\partial \mathbf{f}_t(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_t^t}$$

$$C_t = \left.\frac{\partial \mathbf{g}_t(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_t^{t-1}}$$

Note that $\mathbf{f}_t$ is linearized around $\mathbf{x}_t^t$, while $\mathbf{g}_t$ is linearized around $\mathbf{x}_t^{t-1}$ because $\mathbf{g}_t$ is involved in generating the output $\mathbf{y}_t$.

Substituting (3) and (4) into (1) and (2), we obtain a linear time-varying system with input-like terms

$$\mathbf{x}_{t+1} = (A_t\mathbf{x}_t + \mathbf{d}_t) + \mathbf{w}_t \tag{5}$$

$$\mathbf{y}_t = (C_t\mathbf{x}_t + \mathbf{e}_t) + \mathbf{v}_t \tag{6}$$

where

$$\mathbf{d}_t = \mathbf{f}_t(\mathbf{x}_t^t) - A_t\mathbf{x}_t^t \tag{7}$$

$$\mathbf{e}_t = \mathbf{g}_t(\mathbf{x}_t^{t-1}) - C_t\mathbf{x}_t^{t-1} \tag{8}$$

The goal is to determine $P(\mathbf{x}_t|\{\mathbf{y}\}_1^t)$ and $P(\mathbf{x}_t|\{\mathbf{y}\}_1^T)$ for $t = 1, \ldots, T$. These are the solutions to the filtering and smoothing problems, respectively. Both distributions are normally-distributed for the linearized system, so it suffices to find the mean and variance of each distribution.

We will use the same notation as in [3]. We will simply state the final result or omit certain steps for derivations that correspond exactly to those found in [3].

## 2 Forward Recursions: Filtering

For the linearized system described by (5) and (6), $P(\mathbf{x}_t|\{\mathbf{y}\}_1^t)$ is a normal distribution. We seek its mean $\mathbf{x}_t^t$ and variance $V_t^t$.

$$\begin{aligned}
\log\, P(\mathbf{x}_t|\{\mathbf{y}\}_1^t) &= \log\, P(\mathbf{y}_t|\mathbf{x}_t) + \log\, P(\mathbf{x}_t|\{\mathbf{y}\}_1^{t-1}) + \ldots \\
&= -\frac{1}{2}(\mathbf{y}_t - C_t\mathbf{x}_t - \mathbf{e}_t)' R_t^{-1}(\mathbf{y}_t - C_t\mathbf{x}_t - \mathbf{e}_t) \\
&\quad - \frac{1}{2}(\mathbf{x}_t - \mathbf{x}_t^{t-1})'(V_t^{t-1})^{-1}(\mathbf{x}_t - \mathbf{x}_t^{t-1}) + \ldots \\
&= -\frac{1}{2}\mathbf{x}_t' \left(C_t' R_t^{-1} C_t + (V_t^{t-1})^{-1}\right)\mathbf{x}_t \\
&\quad + \mathbf{x}_t' \left(C_t' R_t^{-1}\mathbf{y}_t - C_t' R_t^{-1}\mathbf{e}_t + (V_t^{t-1})^{-1}\mathbf{x}_t^{t-1}\right) + \ldots
\end{aligned} \tag{9}$$

Using the Matrix Inversion Lemma,

$$\begin{aligned}
V_t^t &= \left(C_t' R_t^{-1} C_t + (V_t^{t-1})^{-1}\right)^{-1} \\
&= V_t^{t-1} - K_t C_t V_t^{t-1}
\end{aligned} \tag{10}$$

where

$$K_t = V_t^{t-1} C_t' \left( R_t + C_t V_t^{t-1} C_t' \right)^{-1}. \tag{11}$$

To find the time update for the variance, we use the fact that $A\mathbf{x}_{t-1}$ and $\mathbf{w}_{t-1}$ are independent and treat $\mathbf{d}_{t-1}$ as a constant

$$\begin{aligned} V_t^{t-1} &= \mathrm{Var}(A_{t-1}\mathbf{x}_{t-1} + \mathbf{d}_{t-1} | \{\mathbf{y}\}_1^{t-1}) + \mathrm{Var}(\mathbf{w}_{t-1} | \{\mathbf{y}\}_1^{t-1}) \\ &= A_{t-1} V_{t-1}^{t-1} A_{t-1}' + Q_{t-1}. \end{aligned} \tag{12}$$

As in [3], we will use the matrix identity

$$(I - (A+B)^{-1}A)B^{-1} = (A+B)^{-1}. \tag{13}$$

Applying (13) and substituting the definition of $\mathbf{e}_t$ from (8),

$$\begin{aligned} \mathbf{x}_t^t &= V_t^t \left( C_t' R_t^{-1} (\mathbf{y}_t - \mathbf{e}_t) + (V_t^{t-1})^{-1} \mathbf{x}_t^{t-1} \right) \\ &= K_t(\mathbf{y}_t - \mathbf{e}_t) + (I - K_t C_t) \mathbf{x}_t^{t-1} \\ &= \mathbf{x}_t^{t-1} + K_t \left( \mathbf{y}_t - \mathbf{g}_t(\mathbf{x}_t^{t-1}) \right). \end{aligned} \tag{14}$$

The time update for the mean can be found by conditioning on $\mathbf{x}_{t-1}$ and substituting the definition of $\mathbf{d}_t$ from (7)

$$\begin{aligned} \mathbf{x}_t^{t-1} &= E_{\mathbf{x}_{t-1}} \left( E(\mathbf{x}_t | \mathbf{x}_{t-1}, \{\mathbf{y}\}_1^{t-1}) \,\middle|\, \{\mathbf{y}\}_1^{t-1} \right) \\ &= E_{\mathbf{x}_{t-1}} \left( A_{t-1}\mathbf{x}_{t-1} + \mathbf{d}_{t-1} \,\middle|\, \{\mathbf{y}\}_1^{t-1} \right) \\ &= A_{t-1} \mathbf{x}_{t-1}^{t-1} + \mathbf{d}_{t-1} \\ &= \mathbf{f}_{t-1}(\mathbf{x}_{t-1}^{t-1}). \end{aligned} \tag{15}$$

The recursions start with $\mathbf{x}_1^0 = \boldsymbol{\pi}_1$ and $V_1^0 = V_1$. Equations (10), (11), (12), (14), and (15) together form the Extended Kalman filter forward recursions, as shown in [1], [2].

## 3 Backward Recursions: Smoothing

Like the filtered posterior distribution $P(\mathbf{x}_t | \{\mathbf{y}\}_1^t)$, the smoothed posterior distribution $P(\mathbf{x}_t | \{\mathbf{y}\}_1^T)$ is also normal. We seek its mean $\mathbf{x}_t^T$ and variance $V_t^T$. We are also interested in the covariance of the joint posterior distribution $P(\mathbf{x}_{t+1}, \mathbf{x}_t | \{\mathbf{y}\}_1^T)$, denoted $V_{t+1,t}^T$.

$$\begin{aligned} \log\, P(\mathbf{x}_{t+1}, \mathbf{x}_t | \{\mathbf{y}\}_1^T) &= \log\, P(\mathbf{x}_{t+1} | \mathbf{x}_t) + \log\, P(\mathbf{x}_t | \{\mathbf{y}\}_1^t) - \log\, P(\mathbf{x}_{t+1} | \{\mathbf{y}\}_1^t) + \log\, P(\mathbf{x}_{t+1} | \{\mathbf{y}\}_1^T) \\ &= -\frac{1}{2} (\mathbf{x}_{t+1} - A_t \mathbf{x}_t - \mathbf{d}_t)' Q_t^{-1} (\mathbf{x}_{t+1} - A_t \mathbf{x}_t - \mathbf{d}_t) \\ &\quad - \frac{1}{2} (\mathbf{x}_t - \mathbf{x}_t^t)' (V_t^t)^{-1} (\mathbf{x}_t - \mathbf{x}_t^t) + \frac{1}{2} (\mathbf{x}_{t+1} - \mathbf{x}_{t+1}^t)' (V_{t+1}^t)^{-1} (\mathbf{x}_{t+1} - \mathbf{x}_{t+1}^t) \\ &\quad - \frac{1}{2} (\mathbf{x}_{t+1} - \mathbf{x}_{t+1}^T)' (V_{t+1}^T)^{-1} (\mathbf{x}_{t+1} - \mathbf{x}_{t+1}^T) + \ldots \\ &= -\frac{1}{2} \mathbf{x}_{t+1}' \left( Q_t^{-1} - (V_{t+1}^t)^{-1} + (V_{t+1}^T)^{-1} \right) \mathbf{x}_{t+1} \\ &\quad - \frac{1}{2} \mathbf{x}_{t+1}' (-Q_t^{-1} A_t) \mathbf{x}_t - \frac{1}{2} \mathbf{x}_t' (-A_t' Q_t^{-1}) \mathbf{x}_{t+1} \\ &\quad - \frac{1}{2} \mathbf{x}_t' \left( A_t' Q_t^{-1} A_t + (V_t^t)^{-1} \right) \mathbf{x}_t + \mathbf{x}_t' \left( -A_t' Q_t^{-1} \mathbf{d}_t + (V_t^t)^{-1} \mathbf{x}_t^t \right) + \ldots \end{aligned} \tag{16}$$

From (16), the covariance matrix of $(\mathbf{x}_{t+1}, \mathbf{x}_t)$ is

$$
\begin{aligned}
\begin{bmatrix} V_{t+1}^T & V_{t+1,t}^T \\ V_{t,t+1}^T & V_t^T \end{bmatrix} &= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} Q_t^{-1} - (V_{t+1}^t)^{-1} + (V_{t+1}^T)^{-1} & -Q_t^{-1}A_t \\ -A_t'Q_t^{-1} & A_t'Q_t^{-1}A_t + (V_t^t)^{-1} \end{bmatrix}^{-1} \qquad (17)
\end{aligned}
$$

Apart from the inclusion of the time subscripts on $A$ and $C$, (18) through (24) are identical to those found in [3], so we will we will simply state the results here.

$$S_{22}^{-1} = V_t^t - J_t V_{t+1}^t J_t', \qquad (18)$$

where we define

$$J_t = V_t^t A_t' (V_{t+1}^t)^{-1}. \qquad (19)$$

$$S_{22}^{-1} S_{21} = -J_t \qquad (20)$$

$$V_t^T = V_t^t + J_t (V_{t+1}^T - V_{t+1}^t) J_t' \qquad (21)$$

$$V_{t+1,t}^T = V_{t+1}^T J_t' \qquad (22)$$

The covariance can also be computed recursively

$$V_{t,t-1}^T = V_t^t J_{t-1}' + J_t (V_{t+1,t}^T - A_t V_t^t) J_{t-1}', \qquad (23)$$

where the recursion is initialized with

$$V_{T,T-1}^T = (I - K_T C_T) A_{T-1} V_{T-1}^{T-1}. \qquad (24)$$

The mean can be found using the same approach as in [3] and substituting the definition of $\mathbf{d}_t$ from (7)

$$
\begin{aligned}
\mathbf{x}_t^T &= -S_{22}^{-1} S_{21} \mathbf{x}_{t+1}^T + S_{22}^{-1} \left( -A_t' Q_t^{-1} \mathbf{d}_t + (V_t^t)^{-1} \mathbf{x}_t^t \right) \\
&= -S_{22}^{-1} S_{21} \mathbf{x}_{t+1}^T + S_{22}^{-1} \left( S_{21} \mathbf{d}_t + (V_t^t)^{-1} \mathbf{x}_t^t \right) \\
&= J_t \mathbf{x}_{t+1}^T - J_t \mathbf{d}_t + (I - J_t A) \mathbf{x}_t^t \\
&= \mathbf{x}_t^t + J_t \left( \mathbf{x}_{t+1}^T - \mathbf{f}_t(\mathbf{x}_t^t) \right). \qquad (25)
\end{aligned}
$$

Equations (19), (21), (23), (24), and (25) together form the Extended Kalman smoother backward recursions. Equivalently, (22) can be used in the place of (23) and (24) to reduce the computation required to find the covariance.

## References


[1] S. Haykin. *Adaptive Filter Theory.* Prentice-Hall, third edition, 1996.

[2] B.D.O. Anderson and J.B. Moore. *Optimal Filtering.* Prentice-Hall, Englewood Cliffs, NJ, 1979.

[3] B.M. Yu, K.V. Shenoy, and M. Sahani. Derivation of Kalman filtering and smoothing equations. 2004.