AMBIGUITY STRUCTURE OF MULTIPATH CHANNELS

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Abstract

Using a recently proposed definition, we characterize the ambiguity function of the multipath underwater acoustic channel for passive localization. Namely, we study the impact of two factors not considered by previous definitions: *(i)* uncertainty about the signal spectrum and *(ii)* existence of multiple paths between the source and the receiver. The importance of accurate channel modeling is addressed by comparing the ambiguity surfaces of methods that: *(i)* use a complete model of the channel; *(ii)* rely only on the information contained in the spatial structure of the incoming wavefield. It is shown that the difference in global behaviour can be explained by a virtual array, whose geometry is determined by the set of temporal inter-path delays.

1 Introduction

Tracking and location systems are being used with increasingly complex models of their environment, such as multiple radiating sources and multipath channels. However, the fast development of signal processing algorithms for these non-trivial situations has not been followed by a corresponding development of global analysis tools. The classical ambiguity of Woodward [8] implicitly admits a set of restrictive assumptions such as perfect knowledge of the statistical description of the observed data for each possible value of the parameters of interest and single source scenarios. Generalizations of Woodward's ambiguity have been presented [7, 2] that allow for the consideration of more general channel models and stochastic narrowband sources. However, the two fundamental limitations referred to above are still present. Forcing the use of the classical definition in more complex scenarios, as for instance passive systems, leads to unrealistic estimates of global performance since the uncertainty about the source signal is not taken into account.

Recently [5, 6, 3], we proposed a new definition of ambiguity function, that is applicable to a wide variety of location problems, in particular, it can be used to analyze the global performance of passive location systems in the presence of an arbitrary number of stochastic sources of unknown spectral characteristics, and in channels exhibiting a complex multipath structure. Based on the Kullback Leibler directed divergence [1], this new measure is motivated by a geometric interpretation of optimal estimators. It recovers Woodward's ambiguity [3] when applied to the active narrowband RADAR problem. For stochastic stationary signals, the ambiguity measure is related to the Itakura-Saito distortion measure, thus allowing the ambiguity analysis to be done directly in terms of the spectral density of the observations.

In this paper, we use the new measure introduced in [5, 3, 6] to study the ambiguity structure of the multipath underwater acoustic channel. In doing so, we study the effect of dropping two hypotheses underlying Woodward's definition: *(i)* knowledge of the source spectrum; *(ii)* constant received power. Dropping hypothesis *(i)* is essential when studying passive systems; condition *(ii)* is incompatible with modeling of the multipath structure of the underwater acoustic channel.

We compare the global performance of systems that use a complete modeling of the channel (both in its temporal and spatial domains) to methods that rely only on the information contained on the spatial structure of the incoming wavefield (the curvature and orientation of the individual wavefronts). We conclude that the information about the source location coded in the set of inter-path delays can considerably improve the global performance. Under certain conditions, this improvement can be interpreted in terms of a virtual array, whose sensors correspond to the multipath arrivals, in a way similar to what has been found when studying the local performance (Cramer-Rao bound) of passive location systems [4].

The paper is organized as follows: In section 2 we present the ambiguity function for location of sources of unknown spectrum in multipath channels. Section 2.1 considers a complete spatial/temporal model, and section 2.2 a purely spatial model. In section 3 we compare the global limit performances of both modeling approaches.

2 Ambiguity Function

Consider that the observations' power spectrum is described by

$$R_\theta(\omega) = \mathcal{S}(\omega) h_\theta(\omega) h_\theta(\omega)^H + \sigma^2(\omega) I_K$$

where we assume that the observation noise is spatially incoherent, with known power density $\sigma^2(\omega)$. In the previous equation, $\mathcal{S}(\omega)$ is the ***unknown*** source spectral density and $h_\theta(\omega)$ is the resultant vector, that describes the coherent combination of the steering vectors corresponding to the P replicas received.

The resultant vector can be decomposed as

$$h_\theta(\omega) = D(\theta) b(\theta)$$

where the $K \times P$ matrix $D(\theta)$ describes the spatial structure of the individual replicas, depending only on the inter-sensor delays for each received path, and $b(\theta)$ is a P dimensional vector that depends only on their temporal alignment.

2.1 Complete Model

When a complete model of the channel is used, the resultant vector is perfectly known for each θ, i.e., both the matrix $D(\theta)$ and the vector $b(\theta)$ in the previous equation are known functions of the source location θ.

In this case, application of the definition of ambiguity introduced in [3, 6] yields the following expression for the ambiguity between scanning location θ, and a source at the true location θ_0 radiating a signal with spectrum $S_0(\omega)$, see [6]:

$$\mathcal{A}(\theta_0, \theta)_{S_0}^{spa/tim} = \int \frac{\mathrm{SNR}(\omega)}{\int \mathrm{SNR}(\omega) d\omega} \mathcal{A}(\theta_0, \theta)_{S_0}^{(c)} - \frac{1}{\int \mathrm{SNR}(\omega) d\omega} \ln \frac{1 + \mathrm{SNR}(\omega) \mathcal{A}(\theta_0, \theta)_{S_0}^{(c)}}{1 + \mathrm{SNR}(\omega)} d\omega \quad (1)$$

where $\mathrm{SNR}(\omega)$ is the ratio of *received* signal to noise power,

$$\mathrm{SNR}(\omega) \triangleq \frac{S_0(\omega) \|h_{\theta_0}(\omega)\|^2}{\sigma^2(\omega)}, \quad (2)$$

and $\mathcal{A}(\theta_0, \theta)_{S_0}^{(c)}$ is the analogue of the classical ambiguity function, i.e., the square of the cosine of the angle between the resultant vectors for the two values of source location.

$$\mathcal{A}(\theta_0, \theta)_{S_0}^{(c)} \triangleq \frac{|h_{\theta_0}(\omega)^H h_\theta(\omega)|^2}{\|h_\theta(\omega)\|^2 \|h_{\theta_0}(\omega)\|^2}.$$

Note that this function can be written using the orthogonal projection operator onto the (one-dimensional) space spanned by the vector $h_\theta(\omega)$:

$$\mathcal{A}(\theta_0, \theta)_{S_0}^{(c)} = \frac{\left\|\Pi_{h_\theta(\omega)} [h_{\theta_0}(\omega)]\right\|^2}{\|h_{\theta_0}(\omega)\|^2}$$

2.2 Spatial Modeling

When the *spatial model* is used, $b(\theta)$ is modeled as an unknown deterministic vector, $b(\omega)$, and the spectral density of the observations has the following form:

$$\mathcal{R}_\theta(\omega) = \sigma^2(\omega) I + \mathcal{S}(\omega) D(\theta) b(\omega) b(\omega)^H D(\theta)^H$$

Simultaneous ignorance of $\mathcal{S}(\omega)$ and $b(\omega)$ implies that only the product $\sqrt{\mathcal{S}(\omega)} b(\omega)$ can be determined, i.e., the only restriction on the noiseless component of $\mathcal{R}_\theta(\omega)$ is that it has rank one, meaning that all the replicas are perfectly correlated. This increased uncertainty leads to [6] the following expression for ambiguity

$$\mathcal{A}(\theta_0, \theta)_{b_0}^{sp} = \int \frac{\mathrm{SNR}(\omega)}{\int \mathrm{SNR}(\omega) d\omega} \Big[\mathcal{A}(\theta_0, \theta)_{b_0}^{(d)} \quad (3)$$

$$- \frac{1}{\mathrm{SNR}(\omega)} \ln \frac{1 + \mathrm{SNR}(\omega) \mathcal{A}(\theta_0, \theta)_{b_0}^{(d)}}{1 + \mathrm{SNR}(\omega)}\Big] d\omega \quad (4)$$

where $\mathrm{SNR}(\omega)$ is defined by eq. (2), and

$$\mathcal{A}(\theta_0, \theta)_{b_0}^{(d)} \triangleq \frac{\left\|\Pi_{\mathcal{H}(\theta)} D(\theta_0) b_0\right\|^2}{\|D(\theta_0) b_0\|^2},$$

and $\Pi_{\mathcal{H}(\theta)}$ denotes the orthogonal projection operator into the subspace $\mathcal{H}(\theta)$, generated by the P steering vectors (columns of the matrix $D(\theta)$) that correspond to the scanning location θ.

Note that in this case the one dimensional vector $h_\theta(\omega)$ is replaced by the P-dimensional subspace spanned by the individual steering vectors. This fact is an immediate consequence of having a larger number of degrees of freedom on the model that is being fitted to the observations.

3 Impact of Temporal Modeling

In the previous section, eq. (1) and eq. (3) differ only on the substitution of $\mathcal{A}(\theta_0,\theta)^{(c)}$ by $\mathcal{A}(\theta_0,\theta)^{(d)}$. Since

$$\mathcal{A} - \frac{1}{\mathrm{SNR}(\omega)} \ln \frac{1+\mathrm{SNR}(\omega)\mathcal{A}}{1+\mathrm{SNR}(\omega)}$$

is a monotonic increasing function of $\mathcal{A}$, we can conclude that comparison of the ambiguity of the two models, $\mathcal{A}(\theta_0,\theta)^{sp/tm}$ and $\mathcal{A}(\theta_0,\theta)^{sp}$, can be done on the basis of the values of $\mathcal{A}(\theta_0,\theta)^{(c)}$ and $\mathcal{A}(\theta_0,\theta)^{(d)}$. From their definitions, and since

$$h_\theta(\omega) \in \mathcal{H}(\theta) \Longrightarrow \Pi_{\mathcal{H}(\theta)} - \Pi_{h_\theta(\omega)} \geq 0$$

we conclude that

$$\mathcal{A}(\theta_0,\theta)^{(d)} \geq \mathcal{A}(\theta_0,\theta)^{(d)}$$

implying

$$\mathcal{A}(\theta_0,\theta)^{sp} \geq \mathcal{A}(\theta_0,\theta)^{sp/tm}$$

i.e., ambiguity is larger when the temporal delays (between paths) are not modeled.

We study now the ambiguity of the two models assuming the following *spatial resolution* conditions:
(i) All the paths corresponding to each source localization (θ_0 and θ) are well resolved by the array:

$$D(\theta)^H D(\theta) \simeq D(\theta_0)^H D(\theta_0) \simeq KI$$

(ii) There are r pairs of paths that fall inside the resolution limit of antenna, such that (see Figure 1):

$$D(\theta_0)^H D(\theta) \simeq K \sum_k e_{i_o(k)} e_{i(k)}^T$$

where e_i denotes a vector whose only component different from zero is the i-th.

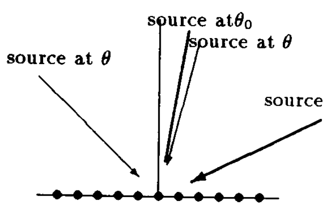

Figure 1: Wavefront geometry.

Let $\breve{b}$ and $\breve{b}_0$ be the vectors that retain only the components of b and b_0 correspondent to the approximately colinear paths:

$$\breve{b} = \left[b_{i(1)} \cdots b_{i(r)}\right], \qquad \breve{b}_0 = \left[b_{i_0(1)} \cdots b_{i_0(r)}\right].$$

Using the two previously presented spatial resolution conditions, the following approximations hold:

$$\mathcal{A}(\theta_0,\theta)^{(d)} \simeq \frac{\|\breve{b}_0\|^2}{\|b_0\|^2}$$

$$\mathcal{A}(\theta_0,\theta)^{(c)} \simeq \frac{\left|\breve{b}_0^H \breve{b}\right|^2}{\|b_0\|^2\|b\|^2} \triangleq \cos^2(\breve{b}_0,\breve{b})$$

where $\cos^2(\breve{b}_0,\breve{b})$ is the cosine of the angle between $\breve{b}_0$ and $\breve{b}$. We see that for the spatial model, whose ambiguity is determined by $\mathcal{A}(\theta_0,\theta)^{(d)}$, ***all the energy received through the unresolved paths*** contributes to increase ambiguity ($\|\breve{b}_0\|^2$).

The previous discussion shows that ambiguity can be considerably smaller for the complete modelization, since the spatially unresolved paths can still be *temporally resolved*, as long as the value of $\cos^2(\breve{b}_0,\breve{b})$ is sufficiently small, or that ***the bandwidth is sufficiently large*** so that the following integral is approximately zero:

$$\int \mathcal{A}(\theta_0,\theta)^{(c)} d\omega = \int \sum_r [\breve{b}_0]_r^* [\breve{b}]_r d\omega.$$

Note that the phase of $[\breve{b}_0]_r^*[\breve{b}]_r$ is determined by the temporal delay between the corresponding paths, showing that the temporal structure of the arrivals can be used to solve ambiguities present in the purely spatial model. $\mathcal{A}(\theta_0,\theta)^{(c)}$ can be written in the form

$$\mathcal{A}(\theta_0,\theta)^{(c)} = \rho \mathcal{A}(\theta_0,\theta)^{(v)}$$

where we defined

$$\rho \triangleq \frac{\|\breve{b}_0\|^2\|\breve{b}\|^2}{\|b_0\|^2\|b\|^2} < 1,$$

and $\mathcal{A}(\theta_0,\theta)^{(v)}$ has the form of $\mathcal{A}(\theta_0,\theta)^{(c)}$ with the resultant vectors $h_{\theta_0}(\omega)$ and $h_\theta(\omega)$ replaced by the vectors $\breve{b}_0$ and $\breve{b}$:

$$\mathcal{A}(\theta_0,\theta)^{(v)} = \frac{|\breve{b}_0^H \breve{b}|^2}{\|\breve{b}_0\|^2\|\breve{b}\|^2}.$$

Using these definitions,

$$\mathcal{A}(\theta_0,\theta)^{sp/tm} = \frac{\int \mathrm{SNR}(\omega)^*}{\int \mathrm{SNR}(\omega)} \overline{\mathcal{A}(\theta_0,\theta)^{(v)}} - \frac{1}{\mathrm{SNR}(\omega)} \int \ln \frac{1+\mathrm{SNR}(\omega)^*}{1+\mathrm{SNR}(\omega)} d\omega$$

where

$$\mathrm{SNR}(\omega)^* \triangleq \rho\,\mathrm{SNR}(\omega),$$

and $\overline{\mathcal{A}(\theta_0,\theta)^{(v)}}$ is given by eq. (1) with $\mathcal{A}(\theta_0,\theta)^{(c)}$ replaced by $\mathcal{A}(\theta_0,\theta)^{(v)}$, i.e., it is the ambiguity for the ***array whose resultant vector is defined by the delays between the spatially unresolved paths.***

For the limit case of no spatial resolution, ($r = P$), i.e.,

$$D(\theta_0)^H D(\theta) \sim KI \Longrightarrow \rho = 1,$$

and

$$\begin{aligned} \mathcal{A}(\theta_0,\theta)^{sp/tm} &\simeq \overline{\mathcal{A}(\theta_0,\theta)^{(v)}} \\ \mathcal{A}(\theta_0,\theta)^{sp} &\simeq 1. \end{aligned}$$

We conclude that ambiguity of the complete model is equal to the virtual array's ambiguity, while the *spatial model is completely ambiguous.*

In the opposite situation, when the antenna spatially resolves all the paths ($r = 0$),

$$\mathcal{A}(\theta_0,\theta)^{(c)} = \mathcal{A}(\theta_0,\theta)^{(d)} \simeq 0$$

and

$$\mathcal{A}(\theta_0,\theta)^{sp} \simeq \mathcal{A}(\theta_0,\theta)^{sp/tm} \simeq \frac{1}{\int \mathrm{SNR}(\omega) d\omega} \int \ln\left(1 + \mathrm{SNR}(\omega)\right) d\omega$$

and the two models have the same ambiguity. The previous expression coincides with the lower limit of ambiguity correspondent to ignorance of the source spectrum, see discussion in [6], showing that there is no room for improvement due to the modelization of the temporal alignment of the received replicas.

As it had been found when studying the local performance, see [4], we conclude that the impact of the temporal structure shows a marked dependency on the characteristics of spatial resolution of the antenna. But, while for the local performance the gains are important specially when there is a good spatial resolution, since the size of the virtual antenna depends on the number of spatially separable rays, from the point of view of the ambiguity behaviour they are a determinant factor when there is small resolving power of the array. Since a good resolving power of the array corresponds already to a good local performance, we may conclude that the *impact of the temporal modelization is specially important from the ambiguity point of view.*

The analogy established here between the gain for the information coded in the set of inter-path delays and a virtual array is not as direct as the one found in the local study [4], we can still conclude that a complete modelization of the received wavefield is important when distinguishing between points that would be otherwise ambiguous if only the spatial dependency of the data was taken into account.

The difference of behaviour of the two modelizations is clear when the conditions that lead to the smallest possible value of ambiguity are considered. While for the temporal/spatial models it is enough to have $D(\theta_0)b_0 \perp D(\theta)b$, which can be verified by many pairs of vectors b_0, b, the spatial modelization requires the stronger condition $\mathrm{Sp}\{D(\theta_0)\} \perp \mathrm{Sp}\{D(\theta)\}$.

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