THE DISCRETE TRIGONOMETRIC TRANSFORMS AND THEIR FAST ALGORITHMS: AN ALGEBRAIC SYMMETRY PERSPECTIVE

Markus Püschel and José M. F. Moura

Department of Electrical and Computer Engineering
Carnegie Mellon University, Pittsburgh, USA

ABSTRACT

It is well-known that the discrete Fourier transform (DFT) can be characterized as decomposition matrix for the polynomial algebra $\mathbb{C}[x]/(x^n - 1)$. This property gives deep insight into the DFT and can be used to explain and derive its fast algorithms. In this paper we present the polynomial algebras associated to the 16 discrete cosine and sine transforms. Then we derive important algorithms by manipulating algebras rather than matrix entries. This makes the derivation more transparent and explains their structure. Our results show that the relationship between signal processing and algebra is stronger than previously understood.

1. INTRODUCTION

There is a large number (several hundred, e.g., [1, 2]) of publications on fast algorithms for the family of 16 discrete trigonometric transforms (DTTs), comprising 8 cosine and 8 sine transforms (DCTs and DSTs). With very few exceptions (including [3, 4, 5]) each of these algorithms has been found by insightful manipulation of the transform matrix entries. We address in this paper two important theoretical questions: (1) Why do these algorithms exist? and (2) How to explain the structure of these algorithms?

To answer these questions, we associate to each DTT a polynomial algebra of the form $\mathcal{A} = \mathbb{C}[x]/p(x)$, with some polynomial p, for which the DTT is a decomposition matrix, i.e., an instantiation of the Chinese remainder theorem (CRT). Then we derive fast algorithms for the DTT by manipulating $\mathcal{A}$, e.g., by a stepwise decomposition or a base change.

In Section 2 we introduce polynomial algebras and associated polynomial transforms and present two general methods for deriving their fast algorithms. Section 3 presents the 16 DTTs and their defining diagonalization properties, which are used in Section 4 to show that the DTTs are polynomial transforms (up to scaling). In Section 5 we then derive important fast DTT algorithms by manipulating their associated algebras.

This work was supported by NSF through award 9988296.

2. MATHEMATICAL BACKGROUND

Polynomial Algebras and Transforms. We denote by $\mathbb{C}[x]$ the algebra of polynomials with complex coefficients. An *algebra* is a vector space that is also a ring, i.e., permits a multiplication that satisfies the distributive law. If p is a polynomial of degree n then we call $\mathcal{A} = \mathbb{C}[x]/p(x)$ a *polynomial algebra*, which is the set of polynomials of degree less than n with multiplication modulo p. If p has pairwise distinct zeros $\alpha = (\alpha_0, \dots, \alpha_{n-1})$, then, by the Chinese remainder theorem, $\mathcal{A}$ decomposes as

$$\mathbb{C}[x]/p(x) \cong \mathbb{C}[x]/(x-\alpha_0) \oplus \ldots \oplus \mathbb{C}[x]/(x-\alpha_{n-1}). \quad (1)$$

If we choose bases $b = (p_0, \dots, p_{n-1})$ in $\mathcal{A}$, and x^0 in $\mathbb{C}[x]/(x - \alpha_k)$, respectively, then the decomposition in (1) is given by the *polynomial transform*

$$\mathcal{P}_{b,\alpha} = [P_\ell(\alpha_k)]_{0 \leq k,\ell < n}. \quad (2)$$

If, more general, we choose bases $a_k x^0$ in $\mathbb{C}[x]/(x - \alpha_k)$, respectively, then we obtain the *scaled* polynomial transform

$$\operatorname{diag}(1/a_0, \dots, 1/a_{n-1}) \cdot \mathcal{P}_{b,\alpha}. \quad (3)$$

Representations. If $q \in \mathcal{A}$, then, because of the distributivity law, the mapping

$$\mathcal{A} \to \mathcal{A}, \quad r \mapsto q \cdot r \quad (4)$$

is linear, and thus represented, w.r.t. the basis b, by a matrix M_q. The mapping

$$\phi : \mathcal{A} \mapsto \mathbb{C}^{n \times n}, \quad q \mapsto M_q \quad (5)$$

is the *(regular) representation* of $\mathcal{A}$ with basis b (and a homomorphism of algebras). As a consequence of (1), for $q \in \mathcal{A}$,

$$\mathcal{P}_{b,\alpha} \cdot \phi(q) \cdot \mathcal{P}_{b,\alpha}^{-1} = \operatorname{diag}(q(\alpha_0), \dots, q(\alpha_{n-1})). \quad (6)$$

From (3) it is clear that any corresponding scaled polynomial transform has the same diagonalization property (6).

Fast Algorithms. We present two general methods for deriving fast algorithms for polynomial transforms $\mathcal{P}_{b,\alpha}$.

0-7803-8116-5/02/$17.00 ©2002 IEEE.

Both methods are based on a decomposition of the algebra $\mathcal{A}$ in (1) *in steps*. We represent the fast algorithms as sparse structured matrix factorizations. In particular, we use the *tensor* or *Kronecker product* of matrices, defined by

$$A \otimes B = [a_{k,\ell} \cdot B], \quad \text{where } A = [a_{k,\ell}],$$

and the *direct sum* of matrices, defined by

$$A \oplus B = \begin{bmatrix} A & \\ & B \end{bmatrix}.$$

An immediate idea for a stepwise decomposition of $\mathcal{A}$ in (1) is to use a factorization $p = q \cdot r$ of p. If β and γ denote the lists of zeros of q and r, respectively, then

$$\begin{aligned} \mathbb{C}[x]/p(x) &\cong \mathbb{C}[x]/q(x) \oplus \mathbb{C}[x]/r(x) && (7) \\ &\cong \bigoplus \mathbb{C}[x]/(x-\beta_i) \oplus \bigoplus \mathbb{C}[x]/(x-\gamma_j) \\ &\cong \bigoplus \mathbb{C}[x]/(x-\alpha_k). && (8) \end{aligned}$$

If we choose c and d as basis of $\mathbb{C}[x]/q(x)$ and $\mathbb{C}[x]/r(x)$, respectively, we get as corresponding factorization

$$\mathcal{P}_{b,\alpha} = P(\mathcal{P}_{c,\beta} \oplus \mathcal{P}_{d,\gamma})B. \quad (9)$$

In particular, B corresponds to (7) and P is the permutation of zeros in step (8).

A more interesting factorization of $\mathcal{P}_{b,\alpha}$ can be derived if $p(x)$ *decomposes* into two polynomials, $p(x) = q(r(x))$, where $\deg(q) = k, \deg(r) = \ell$, i.e., $n = k\ell$. Let $\beta = (\beta_0, \dots, \beta_{k-1})$ be the zeros of q and $\alpha'_i = (\alpha_{i,0}, \dots, \alpha_{i,\ell-1})$ be the zeros of $r(x) - \beta_i$, i.e., each $\alpha_{i,j}$ is a zero α_k of p. Then $\mathcal{A} = \mathbb{C}[x]/p(x)$ decomposes in steps as

$$\begin{aligned} \mathbb{C}[x]/p(x) &\cong \bigoplus \mathbb{C}[x]/(r(x)-\beta_i) && (10) \\ &\cong \bigoplus \bigoplus \mathbb{C}[x]/(x-\alpha_{i,j}) && (11) \\ &\cong \bigoplus \mathbb{C}[x]/(x-\alpha_k). && (12) \end{aligned}$$

To obtain the corresponding factorization of $\mathcal{P}_{b,\alpha}$, we first choose a different basis in $\mathcal{A}$. Let $c = (q_0, \dots, q_{k-1})$ and $d = (r_0, \dots, r_{\ell-1})$ be bases for $\mathbb{C}[x]/q(x)$ and for each $\mathbb{C}[x]/(r(x) - \beta_i)$ in (10), respectively. Then

$$\begin{aligned} b' = (\, & r_0 q_0(r), \dots, r_{\ell-1} q_0(r), \\ & \dots \\ & r_0 q_{k-1}(r), \dots, r_{\ell-1} q_{k-1}(r) \,) \end{aligned} \quad (13)$$

is a basis of $\mathbb{C}[x]/p(x)$. Next, we compute the base change matrix M for step (10) with respect to b' in $\mathcal{A}$ and d in each summand on the right hand side. Because of

$$r_j q_m(r) \equiv r_j q_m(\beta_i) \bmod (r - \beta_i),$$

we get

$$M = [q_j(\beta_i) \cdot \mathrm{I}_\ell]_{0 \le i,j < k} = \mathcal{P}_{c,\beta} \otimes \mathrm{I}_\ell,$$

	closed form	zeros	symmetry
T_n	$\cos(n\theta)$	$\cos\frac{(k+\frac{1}{2})\pi}{n}$	$T_{-n} = T_n$
U_n	$\frac{\sin(n+1)\theta}{\sin\theta}$	$\cos\frac{(k+1)\pi}{n+1}$	$U_{-n} = -U_{n-2}$
V_n	$\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}$	$\cos\frac{(k+\frac{1}{2})\pi}{n+\frac{1}{2}}$	$V_{-n} = V_n$
W_n	$\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$	$\cos\frac{(k+1)\pi}{n+\frac{1}{2}}$	$W_{-n} = -W_n$

Table 1. Four series of Chebyshev polynomials. The range for the zeros is $0 \le k < n$, and $\cos\theta = x$.

where I_ℓ is the $\ell \times \ell$ identity matrix. Step (11) decomposes the summands by polynomial transforms $\mathcal{P}_{d,\alpha'_i}$, respectively, and step (12) reorders the one-dimensional summands with a permutation P. In summary,

$$\mathcal{P}_{b,\alpha} = P\left(\bigoplus_{i=0}^{k-1} \mathcal{P}_{d,\alpha'_i}\right)(\mathcal{P}_{c,\beta} \otimes \mathrm{I}_\ell)B, \quad (14)$$

where B is the base change matrix mapping b to b'.

It is important to note that the factorizations (9) and (14) are useful as fast algorithms only if the matrix B is sparse or can be multiplied with efficiently.

Chebyshev Polynomials. Let $(C_n)_{n\in\mathbb{Z}}$ be a sequence of polynomials that satisfy the three-term recurrence

$$C_n = 2xC_{n-1} + C_{n-2}. \quad (15)$$

We call C_n *Chebyshev polynomials*, and consider the four specific cases $C = T, U, V, W$ arising from the initial conditions $C_0 = 1$ and $C_1 = x, 2x, 2x - 1, 2x + 1$, respectively [6]. Each of these polynomials can be written in closed form, has zeros of the form $\cos r\pi$, $r \in \mathbb{Q}$, and has a symmetry property, see Table 1.

We will need the following properties

$$\begin{aligned} T_m(T_n) &= T_{mn}, && (16) \\ U_{2n-1} &= 2U_{n-1}T_n, && (17) \\ U_{2n} &= V_n W_n. && (18) \end{aligned}$$

3. DISCRETE TRIGONOMETRIC TRANSFORMS

There are 16 discrete trigonometric transforms (DTTs), 8 types of discrete cosine transforms (DCTs) and 8 types of discrete sine transforms (DSTs) [7]. Each of the transforms is given by an $(n \times n)$-matrix M, $n \ge 1$, which multiplies to a signal vector s from the left, $s \mapsto M \cdot s$. As examples, we will use the symbol DCT-2 to refer to a DCT of type 2, DST-7_n to refer to a DST of type 7 and size n.

Table 2 gives the definitions of the 16 DTTs, by stating the respective entry at position (k, ℓ), $0 \le k, \ell < n$, where k is the row index. The definitions given in Table 2 are the *unscaled* versions of the DCTs and DSTs, which will be

	DCTs	DSTs
type 1	$\cos k\ell\frac{\pi}{n-1}$	$\sin(k+1)(\ell+1)\frac{\pi}{n+1}$
type 2	$\cos k(\ell+\frac{1}{2})\frac{\pi}{n}$	$\sin(k+1)(\ell+\frac{1}{2})\frac{\pi}{n}$
type 3	$\cos(k+\frac{1}{2})\ell\frac{\pi}{n}$	$\sin(k+\frac{1}{2})(\ell+1)\frac{\pi}{n}$
type 4	$\cos(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n}$	$\sin(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n}$
type 5	$\cos k\ell\frac{\pi}{n-\frac{1}{2}}$	$\sin(k+1)(\ell+1)\frac{\pi}{n+\frac{1}{2}}$
type 6	$\cos k(\ell+\frac{1}{2})\frac{\pi}{n-\frac{1}{2}}$	$\sin(k+1)(\ell+\frac{1}{2})\frac{\pi}{n+\frac{1}{2}}$
type 7	$\cos(k+\frac{1}{2})\ell\frac{\pi}{n-\frac{1}{2}}$	$\sin(k+\frac{1}{2})(\ell+1)\frac{\pi}{n+\frac{1}{2}}$
type 8	$\cos(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n+\frac{1}{2}}$	$\sin(k+\frac{1}{2})(\ell+\frac{1}{2})\frac{\pi}{n-\frac{1}{2}}$

Table 2. 8 types of DCTs and DSTs (unscaled) of size n. The entry at row k and column ℓ is given for $0 \leq k, \ell < n$.

considered in this paper. The *scaled* versions of the DCTs and DSTs are orthonormal and arise from the unscaled versions by multiplying in some cases the first and/or last row and/or column by $1/\sqrt{2}$, which makes the matrix orthogonal. In addition, the entire matrix is multiplied by a factor to achieve orthonormality. Since we are interested in fast algorithms, it is sufficient to consider the unscaled DTTs.

All 16 DTTs arise as (left) eigenmatrices of certain tridiagonal matrices [8, 9] of size $(n \times n)$, which can be chosen of the form

$$B(\beta_1,\beta_2,\beta_3,\beta_4) = \frac{1}{2}\cdot\begin{bmatrix} \beta_1 & 1 & & & & \\ \beta_2 & 0 & 1 & & & \\ 0 & 1 & 0 & \cdot & & \\ & & 1 & \cdot & 1 & \\ & & & \cdot & 0 & \beta_3 \\ & & & & 1 & \beta_4 \end{bmatrix} \qquad (19)$$

The internal structure of $B(\beta_1,\beta_2,\beta_3,\beta_4)$ corresponds to the equation

$$s_k = \frac{1}{2}(s_{k-1}+s_{k+1}), \quad 1 \leq k \leq n-2. \qquad (20)$$

The entries β_1, β_2 are determined by a choice of *left boundary conditions (b.c.)* that determine how s_{-1} is chosen in (20) for $k = 0$. The 4 left b.c. considered are $s_{-1} = s_1$, $s_{-1} = 0$, $s_{-1} = s_0$, $s_{-1} = -s_0$. For example, the choice $s_{-1} = s_1$ leads to $\beta_1 = 0, \beta_2 = 2$. Similarly, the entries β_3, β_4 are determined by right b.c. arising from the choice of s_n in (20) for $k = n-1$. The right b.c. are the mirrored versions of the left b.c.: $s_n = s_{n-2}$, $s_n = 0$, $s_n = s_{n-1}$, $s_n = -s_{n-1}$. The complete set of values $\beta_1,\beta_2,\beta_3,\beta_4$ for all 16 possible combinations of b.c. is given in Table 3. The DTT associated to each combination of left and right b.c. is given in Table 4. If left and right b.c. are chosen, DTT is the associated transform (from Table 4) and $B(\beta_1,\beta_2,\beta_3,\beta_4)$ the associated tridiagonal matrix (from Table 3), then

$$\text{DTT}\cdot B(\beta_1,\beta_2,\beta_3,\beta_4)\cdot \text{DTT}^{-1} \qquad (21)$$

left b.c.	β_1	β_2	right b.c.	β_3	β_4
$a_{-1} = a_1$	0	2	$a_n = a_{n-2}$	2	0
$a_{-1} = 0$	0	1	$a_n = 0$	1	0
$a_{-1} = a_0$	1	1	$a_n = a_{n-1}$	1	1
$a_{-1} = -a_0$	−1	1	$a_n = -a_{n-1}$	1	−1

Table 3. The values $\beta_1,\beta_2,\beta_3,\beta_4$ from (19) for the 4 respective choices of left b.c. and right b.c.

	$a_n = a_{n-2}$	$= 0$	$= a_{n-1}$	$= -a_{n-1}$
$a_{-1} = a_1$	DCT-1	DCT-3	DCT-5	DCT-7
$= 0$	DST-3	DST-1	DST-7	DST-5
$= a_0$	DCT-6	DCT-8	DCT-2	DCT-4
$= -a_0$	DST-8	DST-6	DST-4	DST-2

Table 4. The left and right boundary conditions associated with the DCTs and DSTs.

is diagonal. As an example, choosing the left b.c. $s_{-1} = s_0$ and the right b.c. $s_n = s_{n-1}$ yields that DCT-2$\cdot B(1,1,1,1)\cdot$ DCT-2^{-1} is diagonal.

4. ALGEBRAS ASSOCIATED TO DTTS

In this section we show that the DTTs are scaled polynomial transforms $D \cdot \mathcal{P}_{b,\alpha}$ (see (3)) by connecting the diagonalization properties of the DTTs in (21) and the diagonalization properties of the (scaled) polynomial transforms in (6). Thus, we construct an algebra $\mathcal{A} = \mathbb{C}[x]/p(x)$ and a basis b, such that for the associated representation ϕ (see (5))

$$\phi(x) = B(\beta_1,\beta_2,\beta_3,\beta_4).$$

In other words, the operation of x (via multiplication) on $\mathcal{A}$ with basis b is reflected by the matrix $B(\beta_1,\beta_2,\beta_3,\beta_4)$. The construction is done in the following three steps.

Internal Structure. All matrices $B(\cdot)$ in (19) have the same internal structure, namely rows $\ldots, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \ldots$. Rewriting (15) as

$$x \cdot C_k = \frac{1}{2}(C_{k-1} + C_{k+1}) \qquad (22)$$

shows that this structure is afforded by any algebra $A = \mathbb{C}[x]/p(x)$, if we choose a basis $b = (C_0,\ldots,C_{n-1})$ ($n = \deg(p)$) of Chebyshev polynomials. In other words, the image of x under the representation ϕ afforded by A with basis b will have an internal structure similar to the matrices $B(\cdot)$.

Left Boundary Conditions. The 4 left b.c. associated with the DTTs are (see Table 4)

$$s_{-1} = s_1,\ s_{-1} = 0,\ s_{-1} = s_0,\ s_{-1} = -s_0. \qquad (23)$$

They apply in the boundary case $k = 0$ in (20). An equivalent behavior is obtained in (22) if we choose the 4 special sequences of Chebyshev polynomials, T_k, U_k, V_k, W_k introduced in Table 1. The symmetry properties of these polynomials (cf. Table 1) correspond to the left b.c. in (23),

$$T_{-1} = T_1,\ U_{-1} = 0,\ V_{-1} = V_0,\ W_{-1} = -W_0, \quad (24)$$

respectively. As an example, *every* algebra $\mathbb{C}[x]/p(x)$ with basis $(T_0, \dots, T_{n-1})$ carries the left b.c. $s_{-1} = s_1$.

Right Boundary Conditions. The 4 right b.c. associated with the DTTs mirror the left b.c. (see Table 4)

$$s_n = s_{n-2},\ s_n = 0,\ s_n = s_{n-1},\ s_n = -s_{n-1}. \quad (25)$$

The right b.c. are determined by the choice of p in $\mathbb{C}[x]/p(x)$. As an example, to introduce the right b.c. $s_n = s_{n-2}$, we choose $p = C_n - C_{n-2}$ where $P \in \{T, U, V, W\}$. Thus the choices of p corresponding to (25) are

$$C_n - C_{n-2}, C_n, C_n - C_{n-1}, C_n + C_{n-1}, \quad (26)$$

respectively.

Example. As an example, we choose DST-3, which has left b.c. b.c. $s_{-1} = 0$ and right b.c. $s_n = s_{n-2}$ (see Table 4). We derive the associated algebra. The left b.c. lead to the basis $b = (U_0, \dots, U_{n-1})$ (see (23) and (24)), and the right b.c. lead to $p = U_n - U_{n-2}$ (see (25) and (26)). Further, $U_n - U_{n-2} = 2T_n$ (using the closed form of U_n and T_n, and trigonometric identities) which has zeros $\alpha_k = \cos((k + 1/2)\pi/n)$, $0 \le k < n$. Thus,

$$\mathcal{P}_{b,\alpha} = [U_\ell(\alpha_k)]_{0 \le k,\ell < n} = D_n \cdot \text{DST-3}_n,$$

with $D_n = \text{diag}_{k=0}^{n-1}(1/(\sin(k + 1/2)\pi/n))$. Equivalently, $\text{DST-3}_n = D_n^{-1} \cdot \mathcal{P}_{b,\alpha}$ is a scaled polynomial transform, as desired.

Summary. The polynomial algebras associated to the 16 DTTs are given in Table 5. Consider a given DTT_n, let $p(x)$ be the polynomial underneath it, and denote with $\alpha_0, \dots, \alpha_{n-1}$ the zeros of p (obtained from Table 1). Further, let f be the associated scaling function (second column) and C the associated type of Chebyshev polynomials (third column). Then

$$\text{DTT}_n = \text{diag}_{k=0}^{n-1}(f(\alpha_k)) \cdot [C_\ell(\alpha_k)]_{0 \le k,\ell < n}$$

is a scaled polynomial transform for $\mathcal{A} = \mathbb{C}[x]/p(x)$. Table 5 also displays the connection with the left and right b.c. (first column and row) identical to Table 4. Further, Table 5 evaluates the sums/differences in (26), which allows to read off the zeros of the polynomial p. It is intriguing that, in a sense, the polynomials $C \in \{T, U, V, W\}$ are *closed* under the operations in (26).

In a different context, [10] recognizes the DTTs of type 1–4 as scaled polynomial transforms without establishing the connection to the b.c.

5. DERIVATION OF FAST ALGORITHMS

Using the algebraic characterization of the DTTs we derive fast algorithms by manipulating the associated polynomial algebra $\mathcal{A}$ rather than the matrix entries of the DTTs. Thus, we show the general principles that account for these algorithms and give insight into their structure. We will repeatedly use Table 5.

Translation of DTTs: Duality. The construction of the algebra $\mathbb{C}[x]/p$ for a given DTT (see Section 4) deals seemingly different with the left b.c. (choice of the base sequence C) and the right b.c. (choice of p). This construction can be reversed and leads to sparse relationships between pairs of DTTs, which we call *dual*.

Definition 1 (Duality) Let DTT and DTT$'$ be at mirrored positions in Table 5, i.e., at positions (i, j), (j, i), $1 \le i, j \le 4$, respectively. We call DTT and DTT$'$ *dual* to each other. The DTTs on the main-diagonal are called self-dual. Dual DTTs have the same associated algebra $\mathbb{C}[x]/p(x)$.

We use the dual pair DCT-3, DST-3 as an example. The algebra $\mathbb{C}[x]/p(x)$ associated to DCT-3_n carries the left b.c. $s_{-1} = s_1$, reflected by the basis $b = (T_0, \dots, T_{n-1})$, and the right b.c. $s_n = 0$, reflected by the equation $p = T_n = 0$. The same b.c. are realized by the *reversed* basis $b' = (U_{n-1}, \dots, U_0)$ (which implies the right b.c. $U_{-1} = 0$) and introducing the left b.c. by the equation $U_n = U_{n-2}$, or $p = U_n - U_{n-2} = 2T_n$. The correspondence between the base polynomials can be displayed as follows; the vertical lines indicate the boundaries.

$$\begin{array}{rlcll} T_{-1} = T_1 & | \ T_0 & \dots & T_{n-1} & | \ T_n = 0 \\ U_n = U_{n-2} & | \ U_{n-1} & \dots & U_0 & | \ U_{-1} = 0 \end{array}$$

In other words, the bases b and b' yield identical representations of $\mathcal{A} = \mathbb{C}[x]/T_n$, and any polynomial transform for $\mathcal{A}$ with basis b is one for $\mathcal{A}$ with basis b', and vice-versa. The algebra $\mathcal{A}$ with basis b is decomposed by DCT-3_n, the algebra $\mathcal{A}$ with basis b' by $\text{DST-3}_n \cdot \text{J}_n$ (J_n is the identity matrix with the columns in reversed order), thus, these matrices are scaled version of each other. And indeed, using trigonometric identities, we find $\text{diag}_{k=0}^{n-1}((-1)^k) \cdot \text{DCT-3}_n = \text{DST-3}_n \cdot \text{J}_n$. Analogous computations verify the same identity for all pairs $\text{DTT}_n, \text{DTT}'_n$ of dual DTTs:

$$\text{diag}_{k=0}^{n-1}((-1)^k) \cdot \text{DTT}_n = \text{DTT}'_n \cdot \text{J}_n. \quad (27)$$

Translation of DTTs: Base Change. Let $\mathbb{C}[x]/p(x)$ be the algebra associated to a given DTT (see Table 5). We observe that the 16 DTTs can be divided into 4 groups of 4 each depending on p being (almost, i.e, up to a linear or quadratic factor) equal to one of T, U, V, W. For example, all DTTs on the main diagonal in Table 5 are in the U-group. We observe that in each row or column there is

	f	C	$s_n - s_{n-2}$	s_n	$s_n - s_{n-1}$	$s_n + s_{n-1}$
$s_{-1} = s_1$	1	T	DCT-1 $2(x^2-1)U_{n-2}$	DCT-3 T_n	DCT-5 $(x-1)W_{n-1}$	DCT-7 $(x+1)V_{n-1}$
$s_{-1} = 0$	$\sin\theta$	U	DST-3 $2T_n$	DST-1 U_n	DST-7 V_n	DST-5 W_n
$s_{-1} = s_0$	$\cos\frac{1}{2}\theta$	V	DCT-6 $2(x-1)W_{n-1}$	DCT-8 V_n	DCT-2 $2(x-1)U_{n-1}$	DCT-4 $2T_n$
$s_{-1} = -s_0$	$\sin\frac{1}{2}\theta$	W	DST-8 $2(x+1)V_{n-1}$	DST-6 W_n	DST-4 $2T_n$	DST-2 $2(x+1)U_{n-1}$

Table 5. Overview of the 16 DTTs and the associated polynomial algebras $\mathbb{C}[x]/p(x)$. The left b.c. (rows) determines a scaling function f ($\cos\theta = x$) and the basis of Chebyshev polynomials $C \in \{T, U, V, W\}$. The right b.c. (columns) then determines the DTT and $p(x)$ (given below the DTT).

exactly one DTT of each group. All 4 DTTs in one group share (almost) the same associated algebra, but with different base sequences $C \in \{T, U, V, W\}$. This connection can be used to translate, by a base change, any two DTTs in the same group into each other using $O(n)$ operations, or, more specifically, using two sparse matrices with $O(n)$ entries each.

As an example, we consider DCT-3_n and DCT-4_n, both in the T-group, i.e., they have the same associated algebra $\mathcal{A} = \mathbb{C}[x]/T_n$, but different bases $b = (T_0, \ldots, T_{n-1})$ and $b' = (V_0, \ldots, V_{n-1})$, respectively. Using $T_\ell = (V_\ell + V_{\ell-1})/2$ (follows also from Table 5) and $V_{-1} = V_0$, the corresponding base change matrix S_n is given by

$$S_n = \frac{1}{2} \cdot \begin{bmatrix} 2 & 1 & & & \\ 0 & 1 & 1 & & \\ & & \cdot & \cdot & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}. \qquad (28)$$

adjusting the scaling factors yields the diagonal matrix $D_n = \operatorname{diag}_{0 \le k < n}(\cos(2k+1)/4n)$ and

$$D_n \cdot \text{DCT-}4_n = \text{DCT-}3_n \cdot S_n, \qquad (29)$$

Transposition or inversion of (29) establishes a sparse relation between DCT-4 and DCT-2. For the special case of dual DTTs, this method leads to relationships different from (27).

Decomposition by Polynomial Factorization. We use (9) to derive a complete set of recursive algorithms in the U-group, i.e., on the main diagonal in Table 5. As an example, we consider the DCT-2_n, $n = 2m$, with associated algebra $\mathcal{A} = \mathbb{C}[x]/(x-1)U_{n-1}(x)$ and basis $b = (V_0, \ldots, V_{n-1})$. Using (17), $(x-1)U_{n-1} = (x-1)U_m T_m$, and thus (compare with (7))

$$\mathbb{C}[x]/(x-1)U_{n-1} \cong \mathbb{C}[x]/(x-1)U_{m-1} \oplus \mathbb{C}[x]/T_m \quad (30)$$

We choose $b' = (V_0, \ldots, V_{M-1})$ as basis in both smaller modules in (30) and determine the base change matrix B_n. From $(x-1)U_{m-1} = (V_m - V_{m-1})/2$ and $T_m = (V_m + V_{m-1})/2$ we derive by induction $V_{m+k} \equiv V_{m-k-1} \bmod (x-1)U_{m-1}$ and $V_{m+k} \equiv -V_{m-k-1} \bmod T_m$, $0 \le k < m$, and get

$$B_n = \begin{bmatrix} \mathrm{I}_m & \mathrm{J}_m \\ \mathrm{I}_m & -\mathrm{J}_m \end{bmatrix}. \qquad (31)$$

The two summands in (30) are decomposed recursively by DCT-2_m and a DCT-4_m, respectively. Reordering the one-dimensional summands with a permutation P_n yields (see (8))

$$\text{DCT-}2_n = P_n(\text{DCT-}2_m \oplus \text{DCT-}4_m)B_n.$$

Analogous derivations yield the full set of recursive algorithms due to (17) and (18). To state the corresponding formulas, we need the following building blocks. The base change matrices B_{2m} in (31) and

$$B_{2m+1} = \begin{bmatrix} \mathrm{I}_m & 0 & \mathrm{J}_m \\ 0 & 1 & 0 \\ \mathrm{I}_m & 0 & -\mathrm{J}_m \end{bmatrix},$$

and the permutation matrices

$$P_{2m} : i \mapsto mi \bmod (2m-1), \quad 0 \le i < 2m-1,$$
$$2m-1 \mapsto 2m-1,$$
$$P_{2m+1} : i \mapsto i(m+1) \bmod (2m+1), \quad 0 \le i \le 2m.$$

Based on $U_{2m-1} = 2U_{m-1}T_m$ we get (e.g., [2, 11])

$$\text{DCT-}1_{2m+1} = P_{2m+1}(\text{DCT-}1_{m+1} \oplus \text{DCT-}3_m)B_{2m+1}.$$
$$\text{DST-}1_{2m-1} = P_{2m-1}(\text{DST-}3_m \oplus \text{DST-}1_{m-1})B_{2m-1}.$$
$$\text{DCT-}2_{2m} = P_{2m}(\text{DCT-}2_m \oplus \text{DCT-}4_m)B_{2m}.$$
$$\text{DST-}2_{2m} = P_{2m}(\text{DST-}4_m \oplus \text{DST-}2_m)B_{2m}.$$

Based on $U_{2m} = V_m W_m$ we get

$$\begin{aligned} \text{DCT-1}_{2m} &= P_{2m}(\text{DCT-5}_m \oplus \text{DCT-7}_m)B_{2m}. \\ \text{DST-1}_{2m} &= P_{2m}(\text{DST-7}_m \oplus \text{DST-5}_m)B_{2m}. \\ \text{DCT-2}_{2m+1} &= P_{2m+1}(\text{DCT-6}_{m+1} \oplus \text{DCT-8}_m)B_{2m+1}. \\ \text{DST-2}_{2m+1} &= P_{2m+1}(\text{DST-8}_{m+1} \oplus \text{DST-6}_m)B_{2m+1}. \end{aligned}$$

We did not find these in the literature.

Decomposition by Polynomial Decomposition. We use (14) to derive algorithms for all transforms in the T-group. As an example we consider DCT-3_n, $n = 2m$. The associated algebra is $\mathbb{C}[x]/T_n$ with basis $b = (T_0, \dots, T_{n-1})$. The zeros of T_n are $\alpha_k = \cos(k + 1/2)\pi/n$, $0 \le k < n$. From (16), $T_n = T_m(T_2)$, and we follow the derivation of (14) to obtain a fast algorithm. It is $q = T_m, r = T_2$ and we choose $c = (T_0, \dots, T_{m-1})$ and $d = (T_0, T_1)$. Thus, b' in (13) is given by $(T_0, T_1, T_2, (T_1+T_3)/2, \dots, T_{n-2}, (T_{n-3}+T_{n-1})/2)$. The base change from b' to b has the structure $P_{2m}(\mathrm{I}_m \oplus S_m)P_{2m}^{-1}$ using S_m from (28) and P_{2m} from the previous paragraph. Thus, the base change from b to b' is given by

$$B_n = P_{2m}(\mathrm{I}_m \oplus S_m^{-1})P_{2m}^{-1}. \tag{32}$$

Further, in (14), $\alpha_i' = (\cos(i+1/2)\pi/n, -\cos(i+1/2)\pi/n)$, and thus

$$\begin{aligned} M_i = \mathcal{P}_{d,\alpha_i'} &= \begin{bmatrix} 1 & \cos(i + 1/2)\pi/n \\ 1 & -\cos(i + 1/2)\pi/n \end{bmatrix} \\ &= \text{DFT}_2 \cdot \text{diag}(1, \cos(i + 1/2)\pi/n). \end{aligned}$$

We get

$$\text{DCT-3}_n = Q_n \left(\bigoplus_{i=0}^{m-1} M_i \right) (\text{DCT-3}_m \otimes \mathrm{I}_2)B_n.$$

with a suitable permutation Q_n. Using $P_{2m}^{-1}(A \otimes \mathrm{I}_2)P_{2m} = \mathrm{I}_2 \otimes A$ for any $m \times m$ matrix A, we obtain the known factorization

$$\begin{aligned} \text{DCT-3}_n = Q_n'(\text{DFT}_2 \otimes \mathrm{I}_m)(\mathrm{I}_m \oplus D_m) \\ (\mathrm{I}_2 \otimes \text{DCT-3}_m)(\mathrm{I}_m \oplus S_m^{-1})Q_n'', \end{aligned}$$

with permutations Q_n', Q_n'' and a diagonal matrix D_m. Inversion or transposition yields an algorithms for DCT-2. The same derivation can be used for each DTT in the T-group if the Chebyshev polynomials in b and c in each case are chosen of the same type $C \in \{T, U, V, W\}$.

Thus, based on $T_{2m} = T_m(T_2)$, we get

$$\begin{aligned} \text{DTT}_n = Q_n'(\text{DFT}_2 \otimes \mathrm{I}_m)(\mathrm{I}_m \oplus D_m) \\ (\mathrm{I}_2 \otimes \text{DTT}_m)(\mathrm{I}_m \oplus S_m^{-1})Q_n'', \end{aligned}$$

for each DTT in the T-group.

6. CONCLUSIONS

We showed that the 16 DTTs can be characterized in the framework of algebras and their representations and used this connection to derive and explain several known DTT algorithms by manipulating algebras rather than DTT matrix entries. We will extend our approach to provide a complete overview of the origin and the derivation of DTT algorithms.

7. REFERENCES

[1] K. R. Rao and P. Yip, *Discrete Cosine Transform: Algorithms, Advantages, Applications*, Academic Press, 1990.

[2] Z. Wang, "Fast Algorithms for the Discrete W Transform and for the Discrete Fourier Transform," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. ASSP-32, no. 4, pp. 803–816, 1984.

[3] G. Steidl and M. Tasche, "A Polynomial Approach to Fast Algorithms for Discrete Fourier-Cosine and Fourier-Sine Transforms," *Mathematics of Computation*, vol. 56, no. 193, pp. 281–296, 1991.

[4] E. Feig and S. Winograd, "Fast Algorithms for the Discrete Cosine Transform," *IEEE Trans. on Signal Processing*, vol. 40, no. 9, pp. 2174–2193, 1992.

[5] S. Egner and M. Püschel, "Automatic Generation of Fast Discrete Signal Transforms," *IEEE Trans. on Signal Processing*, vol. 49, no. 9, pp. 1992–2002, 2001.

[6] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[7] Z. Wang and B.R. Hunt, "The Discrete W Transform," *Applied Mathematics and Computation*, vol. 16, pp. 19–48, 1985.

[8] G. Strang, "The Discrete Cosine Transform," *SIAM Review*, vol. 41, no. 1, pp. 135–147, 1999.

[9] V. Sánchez, P. García, A. M. Peinado, J. C. Segura, and A. J. Rubio, "Diagonalizing Properties of the Discrete Cosine Transforms," *IEEE Trans. on Signal Processing*, vol. 43, no. 11, pp. 2631–2641, 1995.

[10] Th. Kailath and V. Olshevsky, "Displacement structure approach to discrete trigonometric transform based preconditioners of G.Strang and T.Chan type," *Calcolo*, vol. 33, pp. 191–208, 1996.

[11] S.C. Chan and K.L. Ho, "Direct methods for computing discrete sinusoidal transforms," *IEE Proceedings*, vol. 137, no. 6, pp. 433–442, 1990.