Cramér–Rao Bounds for Location of Unknown Multiple Sources in a Multipath Environment

José M. F. Moura* M. João D. Rendas†

*LASIP, Depart. Elect. and Comp. Eng., Carnegie Mellon University, Pittsburgh, PA, 15213.

†CAPS, Depart. Eng. Elect. Comp., Instituto Superior Técnico, 1096 Lisboa Codex, Portugal.

Abstract

We derive expressions for the Cramér-Rao bound on the location parameters of several stochastic sources. The source signals are Gaussian, with unknown mean and variance parameters. In this way, we are able to derive expressions that encompass the commonly considered models of *unknown deterministic* and *zero-mean stochastic* source signals. The inverse of the Cramér-Rao bound for the location parameters is the sum of two components, one representing the information in the mean of the observations, and the other associated to its stochastic nature. Both these components are equal to the information for the corresponding known source signal case minus a loss term, due to lack of knowledge of the source signal moments.

1 Introduction

The localization of sources in complex scenarios, such as multiple sources and/or multipath propagation has deserved the attention of several authors. Two alternative characterizations for the source have been used: **(i)** deterministic unknown signal; **(ii)** stochastic zero-mean Gaussian signal. Description **(i)** is used in [7] where the Cramér-Rao Bound (**CRB**) for the direction of arrival of multiple *narrowband*, *unknown deterministic* sources is derived. We have used description **(ii)** in [6] where we presented the **CRB** for multiple *narrow* or *wideband* zero mean Gaussian sources. Other examples of **CRB** studies are [4,1,2].

The observation vector is Gaussian under both hypotheses. For the unknown deterministic model, it is the mean value of the observations that contains the information regarding the location parameters, while for the stochastic zero mean Gaussian case, it is the spectral density matrix of the observations, or equivalently, its correlation matrix, that depends on the parameters to be estimated. This paper models the source signal as a stochastic Gaussian signal with unknown mean. The **CRB** that we obtain here generalize previously derived expressions yielding as particular cases the **CRB** expressions of [7] and [6].

We begin by presenting a general **CRB** expression for complex Gaussian observations with information in the mean and in the covariance matrix. This expression is then applied to the source location problem, with source signals that are stochastic, wideband, with unknown nonzero means and covariance functions. The inverse of the **CRB** of the location parameters is shown to be the sum of the Fisher Information Matrix (**FIM**) for the stochastic case presented in [6], plus an additional term derived from the mean value of the observations. Both terms are interpreted as the sum of the **FIM** for the active (known signal parameters) case minus a term that represents the information loss due to the estimation of the signal parameters. In both cases, the loss term has the general form of a Gram matrix, in a suitably defined metric.

2 CRB: Unknown Mean and Covariance

In this section, we consider the general case. We observe a K-dimensional vector r which is $\mathcal{N}(\mu, R)$, i.e., a complex Gaussian vector with mean μ and positive–definite covariance matrix R. The probability density function (pdf) is

$$p(r) = (\pi)^{-K}|R|^{-1}\exp\left\{-(r-\mu)^H R^{-1}(r-\mu)\right\}. \quad (1)$$

The mean μ and the covariance R depend on a real unknown deterministic vector θ,

$$\mu = \mu(\theta) \qquad R = R(\theta). \quad (2)$$

It is well known that the **CRB** for θ is given by the inverse of the Fisher Information Matrix (**FIM**) for the vector θ:

$$\mathbf{CRB}(\theta) = \mathcal{F}(\theta)^{-1} \quad (3)$$

where,

$$[\mathcal{F}(\theta)]_{ij} = -\mathrm{E}\left[\frac{\partial^2 \log p(r|\theta)}{\partial\theta_i \partial\theta_j}\right]. \quad (4)$$

Denote the $\log p(r|\theta)$ by $\Lambda(\theta)$. Then,

$$\Lambda(\theta) = -K\log\pi - log|R| - (r-\mu)^H R^{-1}(r-\mu). \quad (5)$$

Following the derivation in [3], where the real case is considered, we obtain the following expression for the generic element of the **FIM** matrix,

$$[\mathcal{F}(\theta)]_{ij} = 2\mathrm{Re}\left\{\frac{\partial\mu}{\partial\theta_i}^H R^{-1}\frac{\partial\mu}{\partial\theta_j}\right\} + \mathrm{tr}\left\{\frac{\partial R}{\partial\theta_i}R^{-1}\frac{\partial R}{\partial\theta_j}\right\}.$$

The **FIM** is the sum of two terms, the first derives from the mean and the second from the covariance. In the

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general case, we have N independent observations which are

$$r_n \sim \mathcal{N}(\mu_n(\theta), R_n(\theta)), \qquad n = 1, \ldots, N. \tag{6}$$

The pdf of the N observations is the product of the individual pdf's. Consequently,

$$\Lambda(\theta) = \sum_{n=1}^{N} \Lambda_n(\theta) \tag{7}$$

where $\Lambda_n(\cdot)$ is the loglikelihood function for each individual observation vector, r_n. From this, it results immediately that

$$\mathcal{F}(\theta) = \sum_{n=1}^{N} \mathcal{F}_n(\theta). \tag{8}$$

Consider now the vector of unknown parameters partitioned into three parts:

$$\theta^T = [\nu^T \ \alpha^T \ \eta^T] \tag{9}$$

such that

$$\mu(\theta) = \mu(\nu, \alpha) \qquad R(\theta) = R(\alpha, \eta). \tag{10}$$

The subcomponents ν and η are signal parameters, while α may collect propagation parameters, (delays), location parameters (e.g., range and depth), and/or channel environmental parameters. From the partitioning above, we can show that the **FIM** has the following form:

$$\mathcal{F}(\nu, \alpha, \eta) = \begin{bmatrix} \mathcal{F}_{\nu\nu} & \mathcal{F}_{\nu\alpha} & 0 \\ \mathcal{F}_{\alpha\nu} & \mathcal{F}_{\alpha\alpha} & \mathcal{F}_{\alpha\eta} \\ 0 & \mathcal{F}_{\eta\alpha} & \mathcal{F}\eta\eta \end{bmatrix} \tag{11}$$

where :

$$[\mathcal{F}_{\nu\nu}]_{ij} = 2\mathrm{Re}\left\{\sum_{n=1}^{N} \frac{\partial \mu_n^H}{\partial \nu_i} R_n^{-1} \frac{\partial \mu_n}{\partial \nu_j}\right\} \tag{12}$$

$$[\mathcal{F}_{\nu\alpha}]_{ij} = 2\mathrm{Re}\left\{\sum_{n=1}^{N} \frac{\partial \mu_n^H}{\partial \nu_i} R_n^{-1} \frac{\partial \mu_n}{\partial \alpha_j}\right\} \tag{13}$$

$$\begin{aligned} [\mathcal{F}_{\alpha\alpha}]_{ij} &= 2\mathrm{Re}\left\{\sum_{n=1}^{N} \frac{\partial \mu_n^H}{\partial \alpha_i} R_n^{-1} \frac{\partial \mu_n}{\partial \alpha_j}\right\} \\ &\quad + \mathrm{tr} \sum_{n=1}^{N} \frac{\partial R_n}{\partial \alpha_i} R_n^{-1} \frac{\partial R_n}{\partial \alpha_j} R_n^{-1} \\ &= [\mathcal{F}_{\alpha\alpha}^{\mu}]_{ij} + [\mathcal{F}_{\alpha\alpha}^{R}]_{ij} \end{aligned} \tag{14}$$

$$[\mathcal{F}_{\alpha\eta}]_{ij} = \mathrm{tr} \sum_{n=1}^{N} \frac{\partial R_n}{\partial \alpha_i} R_n^{-1} \frac{\partial R_n}{\partial \eta_j} R_n^{-1} \tag{15}$$

$$[\mathcal{F}_{\eta\eta}]_{ij} = \mathrm{tr} \sum_{n=1}^{N} \frac{\partial R_n}{\partial \eta_i} R_n^{-1} \frac{\partial R_n}{\partial \eta_j} R_n^{-1} \tag{16}$$

Of interest to us will be the central diagonal block of the inverse of $\mathcal{F}(\nu, \alpha, \eta)$ that determines the **CRB** for the parameters that influence both the mean and the covariance of the observations. It can be shown [3], that

$$\mathbf{CRB}(\alpha) = \mathcal{Q}^{-1} \tag{17}$$

where

$$\mathcal{Q} = \mathcal{F}_{\alpha\alpha} - \mathcal{F}_{\alpha\eta}\mathcal{F}_{\eta\eta}^{-1}\mathcal{F}_{\eta\alpha} - \mathcal{F}_{\alpha\nu}\mathcal{F}_{\nu\nu}^{-1}\mathcal{F}_{\nu\alpha} \tag{18}$$

or

$$\mathcal{Q} = \mathcal{Q}^{\mu} + \mathcal{Q}^{R} \tag{19}$$

where

$$\mathcal{Q}^{\mu} = \mathcal{F}_{\alpha\alpha}^{\mu} - \mathcal{F}_{\alpha\nu}\mathcal{F}_{\nu\nu}^{-1}\mathcal{F}_{\nu\alpha} \tag{20}$$

$$\mathcal{Q}^{R} = \mathcal{F}_{\alpha\alpha}^{R} - \mathcal{F}_{\alpha\eta}\mathcal{F}_{\eta\eta}^{-1}\mathcal{F}_{\eta\alpha}. \tag{21}$$

In the two last equations, the second term in the right hand side represents a loss term, due to the necessity of estimating the signal parameters, the mean in the first case, and the spectral parameters in the second.

3 Multisource Location Problem

3.1 General Model

Each vector of observations is a K-dimensional vector with the following structure:

$$r_n = H_n s_n + w_n, \qquad n = 1, \ldots, N \tag{22}$$

where the complex $K \times S$ matrix H_n depends on the set of parameters α, having the following particular form:

$$H_n = [h_n(\alpha_1) \ldots h_n(\alpha_S)] \tag{23}$$

and $h_n(\alpha_s) \in \mathbf{C}^K, n = 1, \ldots, N; s = 1, \ldots, S$.

In the observations equation, s_n is a sample from an S-dimensional complex Gaussian vector, with mean m_n and covariance matrix S_n, i.e.,

$$s_n \sim \mathcal{N}(m_n, S_n). \tag{24}$$

With the previous assumptions, the observed vector is Gaussian with

$$r_n \sim \mathcal{N}(H_n m_n, H_n S_n H_n^H + \sigma_n I) \tag{25}$$

where we further assumed that the observation noise is a complex, zero mean Gaussian process, with covariance matrix proportional to the identity matrix.

3.2 Location Problem

In the multisource/multipath location problem, that we consider in this paper, the observations at sensor k are given by:

$$r_k(t) = \sum_{s=1}^{S} \sum_{p=1}^{P_s} a_{ksp} s_s(t - \tau_{ksp}) + w_k(t). \tag{26}$$

In this model, S is the number of sources, P_s the number of paths for source s, a_{ksp} and τ_{ksp} the attenuation and delay parameters corresponding to source s along path p to sensor k.

3.3 Narrowband Sources

In the Narrow Band (NB) formalism, the observations r_n are the complex envelopes of the observed vector at a fixed frequency ω. The index n denotes the several snapshots. Using matrix notation,

$$r(t_n) = Hs(t_n) + w(t_n) \tag{27}$$

where the "resultant matrix" H groups the resultant vectors [5] for each of the S sources, that have generic element (at sensor k) given by

$$h(\alpha_s) = \sum_{p=1}^{P_s} a_{ksp} e^{j\omega\tau_{ksp}}. \tag{28}$$

The dependency on the source location vector α_s of the resultant vector $h(\alpha_s)$ is done through the propagation parameters a_{ksp} and τ_{ksp}. Note that in the NB case, the matrix H does not depend on the snapshot index n.

3.4 Wideband Sources

For the wideband (WB) case, where the time×bandwidth product is large, the observations are represented by their DFT coefficients. Under the assumption that the propagation times are small compared to the observation time, the DFT components of the observed signal at frequency ω_n are uncorrelated and given by

$$r(\omega_n) = H(\omega_n)s(\omega_n) + w(\omega_n). \tag{29}$$

In this case, both the resultant matrix H and the source signal DFT coefficients depend upon the index n.

4 The "Mean" Term Q^μ

We analyze in this section the first term Q^μ for the **FIM** in equation (19). This is the only term present when the source signal is deterministic. In fact, the expressions for the deterministic case can be obtained from the expressions in this section, by taking $S_n = 0$.

To proceed, we need to define clearly the meaning of the vectors α and ν. We start with the vector ν that enters in the mean value of the source signal

$$\nu^T = [\nu_1^T \cdots \nu_N^T] \tag{30}$$

where, we recall, N is the number of observation vectors (that represents the number of frequencies for the WB case and the number of snapshots for the NB case). For each frequency, ν_n defines the real and imaginary parts of the complex mean m_n

$$\nu_n^T = [\mathrm{Re}\{m_n\}^T \mathrm{Im}\{m_n\}^T]. \tag{31}$$

Since m_n has dimension S (the number of sources present), each ν_n has dimension $2S$, and ν has dimension $2SN$.

With this partioning of ν, the **FIM** matrix for the mean value parameters is correspondingly partitioned:

$$\mathcal{F}_{\nu\nu} = \begin{bmatrix} \mathcal{F}_{\nu_1\nu_1} & \mathcal{F}_{\nu_1\nu_2} & \cdots & \mathcal{F}_{\nu_1\nu_N} \\ \mathcal{F}_{\nu_2\nu_1} & \mathcal{F}_{\nu_2\nu_2} & \cdots & \mathcal{F}_{\nu_2\nu_N} \\ \vdots & \ddots & \cdots & \vdots \\ \mathcal{F}_{\nu_N\nu_1} & \mathcal{F}_{\nu_N\nu_2} & \cdots & \mathcal{F}_{\nu_N\nu_N} \end{bmatrix}. \tag{32}$$

Using the expression for the mean μ_n, we obtain,

$$\frac{\partial\mu_n}{\partial\mathrm{Re}\{m_m\}} = \delta_{mn}H_m \tag{33}$$

$$\frac{\partial\mu_n}{\partial\mathrm{Im}\{m_m\}} = j\delta_{mn}H_m, \tag{34}$$

from where we may conclude that the non-diagonal blocks in $\mathcal{F}_{\nu\nu}$ are zero, and the diagonal blocks are

$$\mathcal{F}_{\nu_n\nu_n} = \begin{bmatrix} \mathrm{Re}\{X_n\} & -\mathrm{Im}\{X_n\} \\ \mathrm{Im}\{X_n\} & \mathrm{Re}\{X_n\} \end{bmatrix} \tag{35}$$

where we defined

$$X_n = H_n^H R_n^{-1} H_n. \tag{36}$$

Let Y_n denote the inverse of X_n. Then, [7], the inverse of $\mathcal{F}_{\nu\nu}$ is also a block diagonal matrix, with diagonal block

$$\mathcal{F}^{-1}_{\nu_n\nu_n} = \begin{bmatrix} \mathrm{Re}\{Y_n\} & -\mathrm{Im}\{Y_n\} \\ \mathrm{Im}\{Y_n\} & \mathrm{Re}\{Y_n\} \end{bmatrix}. \tag{37}$$

The partioning of the mean value vector ν induces also an identical partioning in the cross term $\mathcal{F}_{\alpha\nu}$:

$$\mathcal{F}_{\alpha\nu} = [\mathcal{F}_{\alpha\nu_1} \ \ldots \ \mathcal{F}_{\alpha\nu_N}]. \tag{38}$$

We may now write for the loss term in Q^μ:

$$\mathcal{F}_{\alpha\nu}\mathcal{F}_{\nu\nu}^{-1}\mathcal{F}_{\nu\alpha} = \sum_{n=1}^{N} \mathcal{F}_{\alpha\nu_n}\mathcal{F}^{-1}_{\nu_n\nu_n}\mathcal{F}_{\nu_n\alpha}. \tag{39}$$

It is easily verified that

$$\frac{\partial\mu_n}{\partial\alpha} = D_n M_n \tag{40}$$

where we defined $M_n \triangleq \mathrm{diag}\{m_n\}$ and D_n is the matrix formed by the derivatives of each resultant vector h_n with respect to its location parameter:

$$D_n = \left[\frac{\partial h_n(\alpha_1)}{\partial\alpha_1} \ \cdots \ \frac{\partial h_n(\alpha_S)}{\partial\alpha_s}\right]. \tag{41}$$

These expressions yield the following

$$\mathcal{F}_{\alpha\nu_n} = 2\,[\mathrm{Re}\{\Delta_n\} \ \mathrm{Im}\{\Delta_n\}] \tag{42}$$

where

$$\Delta_n \triangleq M_n^* D_n^H R_n^{-1} H_n, \tag{43}$$

and

$$\mathcal{F}^\mu_{\alpha\alpha} = 2\sum_{n=1}^{N} \mathrm{Re}\left\{M_n^* D_n^H R_n^{-1} D_n M_n\right\}. \tag{44}$$

Finally, we obtain

$$\mathcal{F}_{\alpha\nu}\mathcal{F}_{nu\nu}^{-1}\mathcal{F}_{\nu\alpha} = \sum_{n=1}^{N} \mathrm{Re}\{\Delta_n Y_n \Delta_n^H\} \quad (45)$$

and

$$\mathcal{Q}^{\mu} = 2 \sum_{n=1}^{N} \mathrm{Re}\left\{M_n^* D_n^H R_n^{-1} \left[R_n - H_n(H_n^H R_n^{-1} H_n)^{-1} H_n^H\right] R_n^{-1} D_n M_n\right\}. \quad (46)$$

The **CRB** expression in [7] is for unknown deterministic signals. Correspondingly, the results in [7] are obtained from equation (46) when we make the source covariance matrix zero, i.e., $S_n = 0$. Equivalently, eq. (46) says that formally the results in [7] correspond to the "mean" term $\mathcal{Q}^{\mu}$ here studied as long as the inner products are taken not in the covariance matrix R_n but in the noise only covariance matrix $R_n = \sigma_n I$. The loss term in this expression, arising from the estimation of the signal amplitudes, is the real part of a Gram matrix. Consider the Hilbert space of complex K-dimensional sequences with the following defintion of inner-product: $\langle f, g\rangle_{H(R_n)} = f^H R_n^{-1} g$ where R_n is a definite positive $K \times K$ Hermitian matrix. In this Hilbert space R_n is the identity operator, and $P = H_n(H_n^H R_n^{-1} H_n)^{-1} H_n^H$ is the orthogonal projection operator in the space spanned by the columns of H_n. It is easily verified that $P_n[P_n] = P_n^H R_n^{-1} P_n = P_n$ and that the loss term (45) can be written as

$$2\mathrm{Re}\left\{\sum_n \left\langle P_n\left[\frac{\partial \mu_n}{\partial \alpha}\right], P_n\left[\frac{\partial \mu_n}{\partial \alpha}\right]\right\rangle_{H(R_n)}\right\}.$$

The inner expression is the Gram matrix of the projection of the derivative of the mean value with respect to the location parameters in the space spanned by the resultant vectors, taken in the Hilbert space $H(R_n)$.

The influence of the multipath structure and of the array geometry could be analyzed here, considering a decomposition of the resultant vectors for each source in terms of the "steering vectors" for each multipath arrival and the "multipath delays", measured at a reference sensor. Since the expression for the deterministic signal is similar to the expression for the stochastic signal at high signal-to-noise ratios, see [5], the conclusions presented therein remain valid. We do not pursue this study here.

5 The "Covariance" Term $\mathcal{Q}^R$

In this section we present the expressions for the covariance based term $\mathcal{Q}^R$, for the case of S propagating uncorrelated sources, i.e, when

$$S_n = \mathrm{diag}\{\rho_{ns}\}. \quad (47)$$

For each source, the sequence $\rho_{ns}, n = 1, \ldots, N$ is parametrized by the L-dimensional vector η_s:

$$\rho_{ns} = \rho_{ns}(\eta_s). \quad (48)$$

The complete vector of unknown spectral parameters is the SL-dimensional vector:

$$\eta^T = \left[\eta_1^T \cdots \eta_S^T\right]. \quad (49)$$

The first term in (21), $\mathcal{F}_{\alpha\alpha}^R$, is the **FIM** for the case of known source spetral parameters. The case of multiple propagating sources with arbitrary known covariance matrix has been treated in detail in [5], where the following expression was derived:

$$\mathcal{F}_{\alpha\alpha}^R = 2 \sum_{n=1}^{N} \mathrm{Re}\left\{(S_n H_n^H R_n^{-1} H_n S_n) \odot (D_n^H R_n^{-1} D_n) + (D^H R_n^{-1}) \odot (D^H R_n^{-1})\right\}. \quad (50)$$

In this expression, $\odot$ denotes the Hadamard (element by element) product of matrices.

To compute the other two terms in (21), we note that,

$$\frac{\partial R_n}{\partial \eta_{s_l}} = \frac{\partial \rho_{ns}}{\partial \eta_{s_l}} h_n(\alpha_s) h_n(\alpha_s)^H \quad (51)$$

and

$$\frac{\partial R_n}{\partial \alpha_s} = \rho_{ns}\left[\frac{\partial h_n(\alpha_s)}{\partial \alpha_s} h_n(\alpha_s)^H + h_n(\alpha_s)\frac{\partial h_n(\alpha_s)}{\partial \alpha_s}^H\right].$$

Using these expressions, we obtain,

$$\mathcal{F}_{\eta\eta} = \sum_{n=1}^{N} \frac{\partial \rho_n^T}{\partial \eta}\left[(H_n^H R_n^{-1} H_n) \odot (H_n^H R_n^{-1} H_n)^T\right] \frac{\partial \rho_n}{\partial \eta}$$

where $\dot{h}_{n\alpha}$ denotes $\partial h_n/\partial \alpha$, and $\partial \rho_{ns}/\partial \eta_s$ is the L-dimensional row vector of derivatives of the spectral level of source s, ρ_{ns}, with respect to the L spectral parameters.

For the cross-term, we obtain the following expression

$$\mathcal{F}_{\alpha\eta} = 2\mathrm{Re}\left\{\sum_{n=1}^{N} S_n \left[(H_n^H R_n^{-1} H_n) \odot (H_n^H R_n^{-1} D_n)^T\right] \frac{\partial \rho_n}{\partial \eta}\right\}.$$

These expressions can now be used in (21) to determine $\mathcal{Q}^R$. The resulting expression is difficult to analyze. To obtain more insight into the problem, we consider the simpler case of a single propagating source, $S = 1$. In this case, the inverse of the covariance matrix R_n is easily found using the matrix inversion lemma. Define the scalar quantity $E_n = \sigma_n + \rho_n \|h\|^2$, and let v_i be the N-dimensional vector of generic element

$$[v_i]_n = \frac{\rho_n}{E_n}\mathrm{Re}\left\{\frac{\partial h_n}{\partial \alpha_i}^H h_n\right\}. \quad (52)$$

Define the $N \times L$ matrix

$$[\phi]_{nl} = \frac{\|h_n\|^2}{E_n}\frac{\partial S_n}{\partial \eta_l}. \quad (53)$$

It can be shown that $\mathcal{F}_{\eta\eta}$ is the $(L \times L)$ Gram matrix of the columns of this matrix, $\{\phi_i\}_{i=1}^{L}$,

$$\mathcal{F}_{\eta\eta} = \langle \phi^T, \phi \rangle . \quad (54)$$

which we assume non-singular.

Define the the matrix P,

$$P = \phi \langle \phi^T, \phi \rangle^{-1} \phi^T . \quad (55)$$

P is the orthogonal projection matrix into $\mathcal{S}_\phi$, the space spanned by the N-dimensional vectors $\{\phi_i\}_{i=1}^{L}$. This subspace has dimension L.

The loss term in $\mathcal{Q}^R$ is the Gram matrix of the projection of the vectors v_i in $\mathcal{S}_\phi$,

$$\mathcal{F}_{\alpha\eta}\mathcal{F}_{\eta\eta}^{-1}\mathcal{F}_{\eta\alpha} = \langle P[v]^T, P[v] \rangle \quad (56)$$

With these expressions, it can be shown [5], that $\mathcal{Q}^R$ can be written as

$$\mathcal{Q}^R = \mathbf{CRB}(\alpha)_{unk}^{-1} + \mathcal{G}(\alpha, \eta) \quad (57)$$

where $\mathbf{CRB}(\alpha)_{unk}^{-1}$ is the **FIM** for completely unknown spectrum, given by

$$\mathbf{CRB}(\alpha)_{unk}^{-1} = \sum_n K_1(n) \mathrm{Re}\left\{ \dot{h}_{n\alpha}^{H} P_h^{\perp}(n) \dot{h}_{n\alpha} \right\} . \quad (58)$$

where $P_h^{\perp}(n)$ denotes the orthogonal projection matrix in the orthogonal complement of the span of the vector h_n, $K_1(n) = S_n^2 \|h\|^2 / S_n E_n$, and $\mathcal{G}$ is a gain term that represents the additional information due to the known parametrization of the source spetrum,

$$\mathcal{G}(\alpha, \eta) = \left\langle P_{\mathcal{S}_\phi}^{\perp}[v_n], P_{\mathcal{S}_\phi}^{\perp}[v_n]^T \right\rangle . \quad (59)$$

where $P_{\mathcal{S}_{\phi_L}}^{\perp}$ is the projection operator in the orthogonal complement of $\mathcal{S}_{\phi_L}$.

We analyze two extreme cases:

(i) No information gain: $\mathcal{G} = 0$. to have $\mathcal{G} = 0$, we must require that all the vectors v_i belong to $\mathcal{S}_\phi$, i.e., there must exist vectors t_i such that

$$v_i = \phi t_i . \quad (60)$$

Note that if for a given subset of the unknown parameters this condition is satisfied, then it will be trivially satisfied for the complete vector η, showing that having additional unknown parameters cannot remove ambiguities, as it should be.

If $N = L$, as in the case of arbitrary source spectrum, $\mathcal{S}_\phi$ has full dimension N, and its orthogonal complement is trivially equal to the zero vector. In this case, (60) is trivially satisfied, and $\mathcal{G} = 0$ always.

(ii) No information loss. The condition for no information loss is that all the vectors v_i belong to $\mathcal{S}_\phi^{\perp}$. Consider a fixed spectral parameter vector η_0 of dimension L_0. If we add another unknown parameter to form η_{L_0+1}, the subspace $\mathcal{S}_{\phi_{L_0+1}}^{\perp}$ is a proper subspace of $\mathcal{S}_{\phi_{L_0}}^{\perp}$, and consequently $\mathcal{G}_{L_0} \geq \mathcal{G}_{L_0+1}$, as it should be expected.

We note that the expression of $\mathbf{CRB}(\alpha)_{unk}$ shows that for $K = 1$, that is, when there is a single observation sensor, $P_h^{\perp} = 0$ and no estimation is possible. This fact is intuitively justified since in this case the estimation is based on a "covariance fitting" mechanism, that determines the multipath delays that are more likely to have transformed the source signal spectrum into the observed spectrum. If no source spectral information is available, this fitting operation is not possible.

6 Multipath Structure

The detailed analysis of the impact of the multipath structure is difficult to carry out in the general multiple source context. For a single source, we have shown [5] that the contribution due to the multipath structure is highly dependent on the resolving power of the array for the several incoming paths. Namely, the multipath contribution can be interpreted as the contribution of a "virtual array", whose "size" and "geometry" are dependent on the beampattern of the physical array for the several incoming wavefronts. In particular, the "size" of this virtual array is equal to the number of "clusters" of rays resolved by the array.

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