Complex Numbers

Definition:

$\mathbb{C} = \{ x + jy ; x, y \in \mathbb{R}, j = \sqrt{-1} \}$

$(\mathbb{C}, +, \cdot)$ is a field, 2D vector space over $\mathbb{R}$, basis $\{1, j\}$

- We extend the reals by j, the solution to $z + 1 = 0$

  $j = \sqrt{-1}$

- over $\mathbb{C}$, $z^2 = s$, $\forall s \in \mathbb{R}$ and

  $z^2 = c$ $\forall c \in \mathbb{C}$ can be solved

- $\mathbb{C}$ is closed wrt $\sqrt{\cdot}$

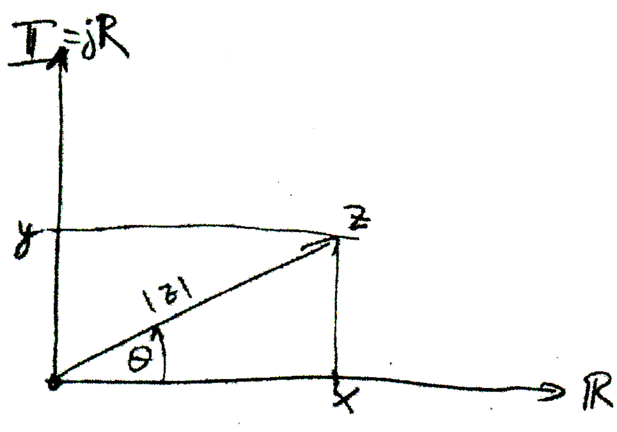
- Physics/Math: $\sqrt{-1} = i$, ECE: $\sqrt{-1} = j$

Real / Imaginary part

$x = Re(z)$, $y = Im(z)$ $\Longleftrightarrow$ $z = x + jy$ $\mathcal{R}$, $\mathcal{I}$

Rectangular format: $x + jy$

e.g. $(2+j) + 3j$ is NOT in rect format



Polar representation

$z = x + jy \rightarrow (r, \theta)$

$r = |z|$

$\theta = \measuredangle z$ "argument" or "phase"

$= \arg z$

However: $\arg z = \text{Arg}\, z + 2k\pi = \{\theta_0, \theta_0 \pm 2\pi, \theta_0 + 4\pi, \dots\}$

$\uparrow$

$PV(\arg z)$: $[-\pi, \pi)$, can be chosen $(-\pi, \pi]$, $[0, 2\pi)$

$\arg(\cdot)$ is a "multivalued function", countable many values

Remark: finding the phase (unwrapping it) is a common challenge in signal processing

→ $\boxed{\text{go to (3) Examples here}}$

Polar ⟺ Rectangular

for $z = x + jy$:

$r = |z| = +\sqrt{x^2 + y^2} \geq 0$

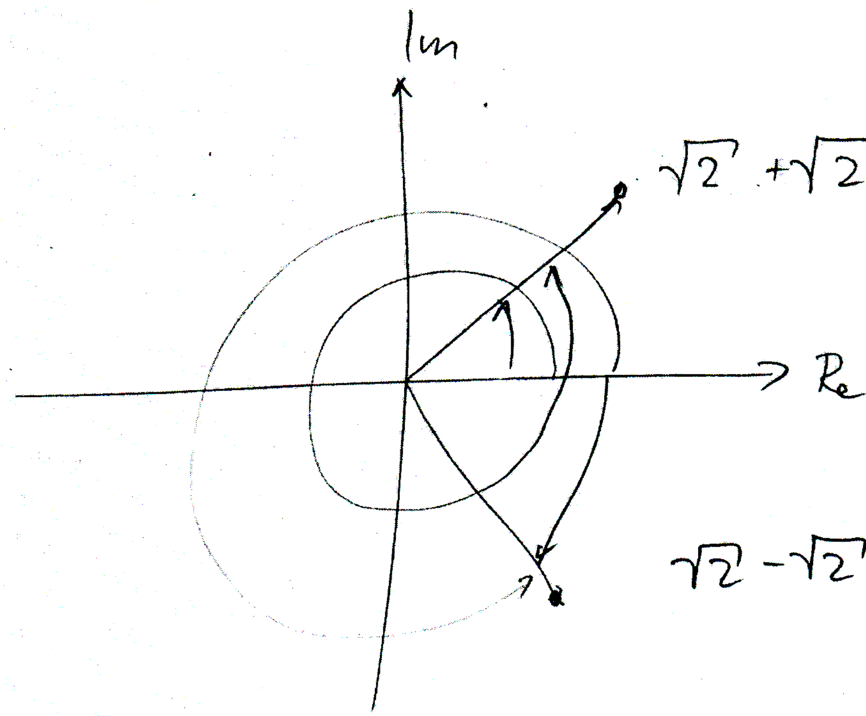
$\arg z = \tan^{-1} \frac{y}{x}$

for $z = (r, \theta)$:

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

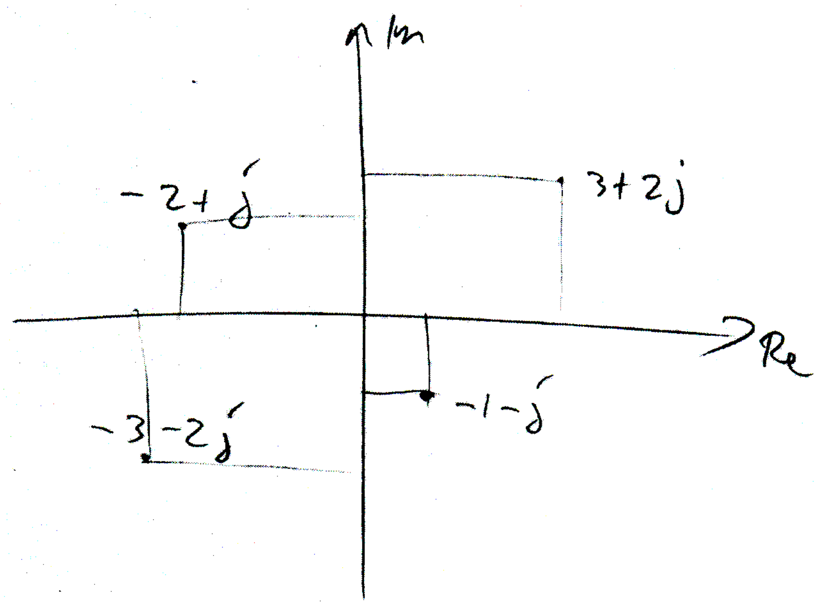
Polar format examples



$$\sqrt{2}+\sqrt{2}j = (2, 45^\circ) = (2, 405^\circ) = \left(2, \frac{\pi}{4}\right) = \left(2, \frac{9\pi}{4}\right) \ \vdots$$

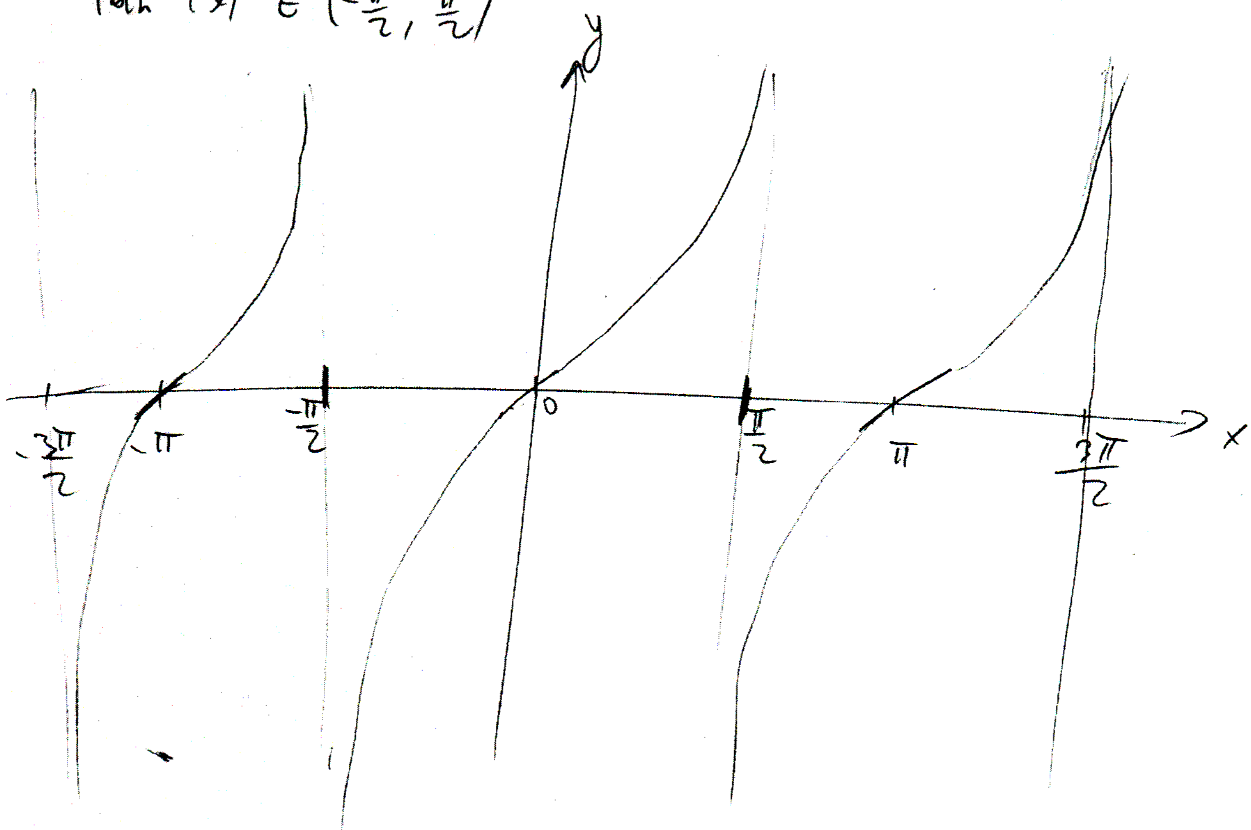
$$\sqrt{2}-\sqrt{2}j = (2, -45^\circ) = (2, 315^\circ) = \left(2, -\frac{\pi}{4}\right) = \left(2, \frac{7\pi}{4}\right) \ \vdots$$

Rectangular Examples



Arg and PV of $\tan^{-1}$

- $\tan^{-1}(\cdot)$ is inverse of $\tan(\cdot)$
- multi-valued function with countable many numbers
- $\tan^{-1}(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$



$\Rightarrow$ need to look at sign of x and y to know quadrant

we use $\tan^{-1}\left(\frac{y}{x}\right) = \operatorname{Arg} z \in [-\pi, \pi)$

PV

$\operatorname{Arg} z = \arg z \bmod 2\pi = \arg z + \underbrace{2\pi \left\lceil -\frac{1}{2} - \frac{\arg z}{2\pi} \right\rceil}$

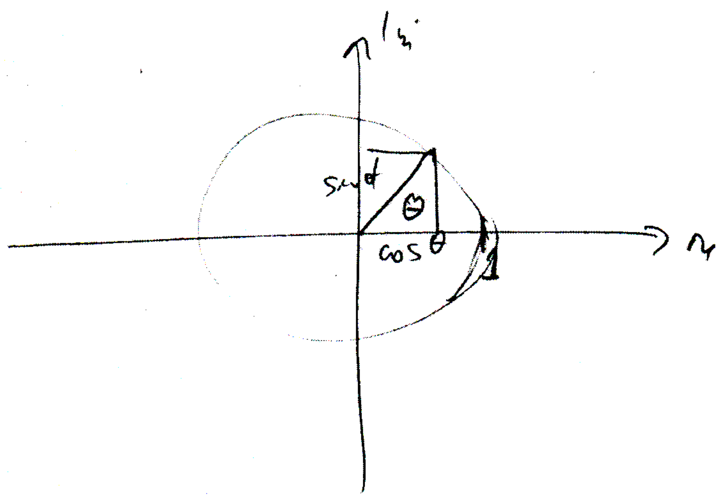
$\lceil \cdot \rceil$ ceiling function (smallest integer larger than number)

$\lceil x \rceil = m :\Leftrightarrow m \in \mathbb{Z},\ m \geq x$

$\lceil -3 \rceil = -3$, $\lceil -3.5 \rceil = -3$; $\lceil 4.25 \rceil = 5$

Exponential representation

- periodicity of $\measuredangle z$ "wraps" the argument around the unit circle



- Euler Formula:

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad , \quad |e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$z = r\cos e^{j\theta}$$

- careful : $r \geq 0$

if not : $z = |a| e^{j(\theta - \pi)}$ as $-1 = e^{-\pi j}$

Exponential $\rightarrow$ Polar

$|z| = |ae^{j\theta}| = \sqrt{(a\cos^2\theta) + a\sin^2\theta|} = |a|$

$\arg z = \theta + 2\pi k, \; k \in \mathbb{Z}$ for $a > 0$

if $a < 0$:
$\arg z = \theta + \pi(2k+1), \; k \in \mathbb{Z}$

Polar and Exponential representation are closely related: $(r, \theta) \Leftrightarrow re^{j\theta}$

Exponential $\leftrightarrow$ Rectangular

$$z = x + jy$$

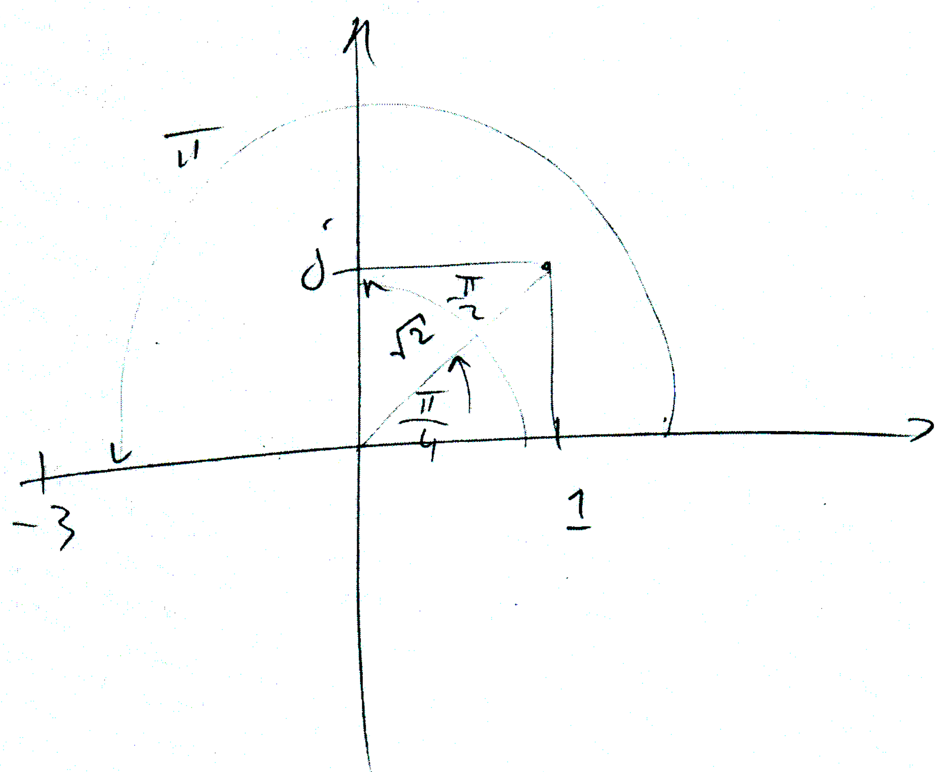
$$\Rightarrow z = \sqrt{x^2+y^2}\, e^{j\tan^{-1}\frac{y}{x}}$$

Examples

$$1+j = \sqrt{2}\, e^{j\frac{\pi}{4}}$$

$$-3 = -3 e^{j\pi}$$

$$j = e^{j\frac{\pi}{2}}$$



Arithmetic with complex numbers

Equality

$z_1 = z_2 \iff \mathrm{Re}\, z_1 = \mathrm{Re}\, z_2 \wedge \mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$

$\iff |z_1| = |z_2| \wedge \arg z_1 = \arg z_2$

Example: $1 + j = \sqrt{2} \cos\frac{\pi}{4} + j\sqrt{2} \sin\frac{\pi}{4}$

Complex Conjugation

for $z = x + jy$: $z^* = x - jy$ rect

$z = re^{j\theta}$: $z^* = re^{-j\theta}$ exp.

$z = (r, \theta) \Rightarrow z^* = (r, -\theta)$ polar

Remark: conjugating a complex number in polar/exp. format:

$\mathrm{Arg}\, z \to -\mathrm{Arg}\, z$

$-\arg z = -(\mathrm{Arg}\, z + k2\pi) = \begin{cases} -\mathrm{Arg}\, z + 2k\pi, & \mathrm{Arg}\, z > -\pi \\ \mathrm{Arg}\, z + 2k\pi, & \mathrm{Arg}\, z = \pi \end{cases}$

special case

Remark: $z^* = z$ for $z \in \mathbb{R}$

$z^* = -z$ for $z \in j\mathbb{R}$

Addition

Addition only works in rectangula ~~format~~:

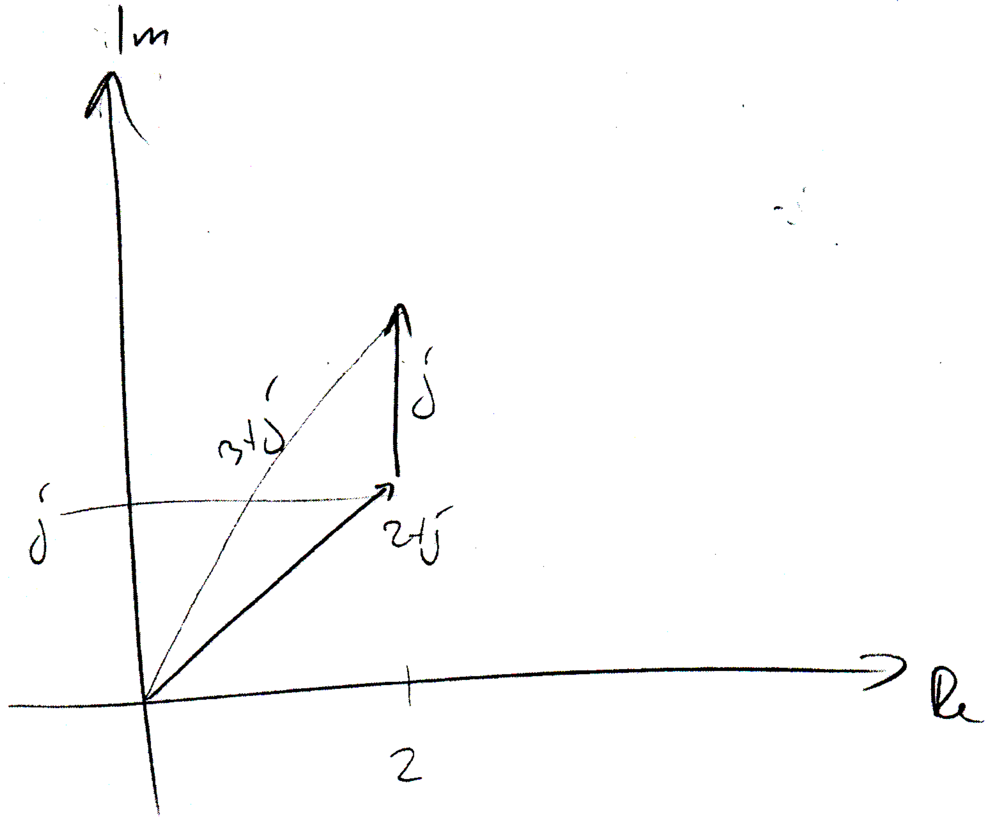
$$z_1 = x_1 + j\,y_1 \qquad z_2 = x_2 + j\,y_2 \implies z_1 + z_2 = (x_1 + x_2) + j\,(y_1 + y_2)$$

If in another format, first convert

$$z_1 = e^{j\pi} = -1 \qquad z_2 = e^{j\frac{\pi}{2}} = j \implies z_1 + z_2 = -1 + j$$

$\mathbb{C}$ is a 2D vector space over $\mathbb{R}$ with basis $\{1, j\}$

$\Rightarrow$ addition is vector addition



Multiplication

Multiplication can be done in rect format but is easier in Exp f.

rect format

$$z_1 = x_1 + jy_1 \qquad z_2 = x_2 + jy_2$$

$$\Rightarrow z_1 \cdot z_2 = (x_1 + jy_1)\cdot(x_2 + jy_2) =$$

$$= x_1x_2 + jx_1y_2 + jy_1x_2 + j^2 y_1y_2 =$$

$$= \underline{(x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)}$$

standard assoc, dist, comm rules apply as $\mathbb{C}$ is a field and $\mathbb{R} \supseteq \mathbb{C}$

Exp format:

$$z_1 = r_1 e^{j\theta_1} \qquad z_2 = r_2 e^{j\theta_2}$$

$$\Rightarrow z_1 \cdot z_2 = r_1 e^{j\theta_1} \, r_2 e^{j\theta_2}$$

$$= r_1 r_2 e^{j\theta_1} e^{j\theta_2}$$

$$= (r_1 r_2) e^{j(\theta_1 + \theta_2)}$$

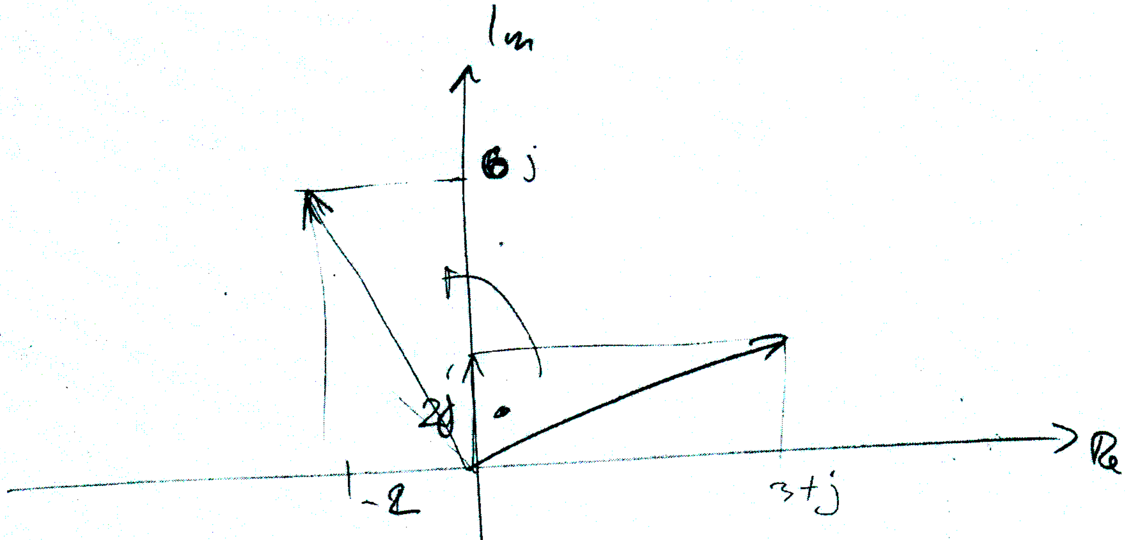
Remark:

Multiplication is "stretch + rotate"

- if $z_2 \in \mathbb{R} \Rightarrow$ only stretch
- if $|z_2| = 1 \Rightarrow$ only rotate

Note that z_1 rotates/stretches z_2 and z_2 rotates/stretches z_1

Therefore very useful for ECE



$(3+j)\cdot(2j) = -2+6j$